Lecture 1

Recap :- Mechanics of materials.

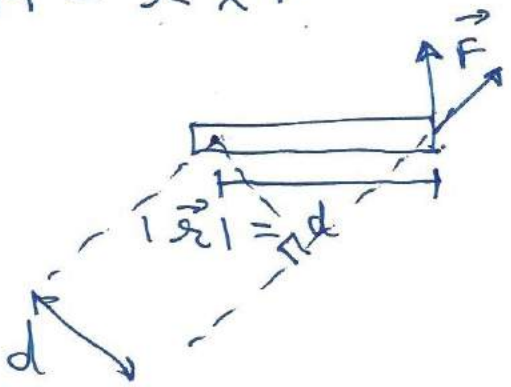
(i) Static i.e. Equilibrium

$P_i$ ; F

$\vec{F} = F_x \vec{x} + F_y \vec{y} + F_z \vec{z}$

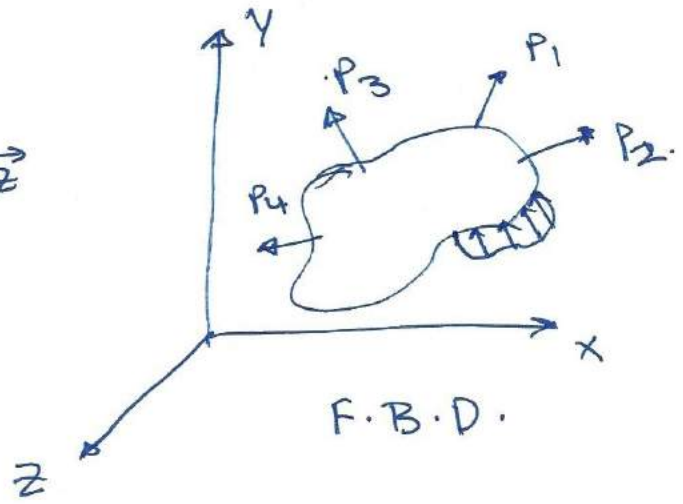
3-D.

$\vec{M} = \vec{r} \times \vec{F}$




$|\vec{r}| = d$

$M = F \cdot d$




F.B.D.

(1) Equilibrium

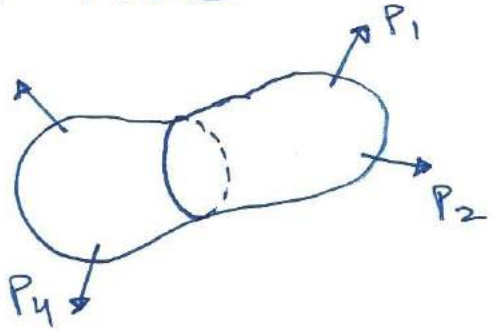
$$\sum_{i=1}^{3} F_{x_i} = F_{x_1} + F_{x_2} + F_{x_3} = 0$$

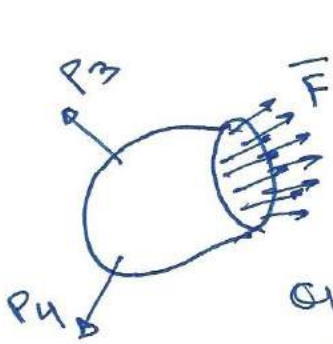
$$\Sigma F_y = 0 \; ; \; \Sigma F_z = 0$$

$$\Sigma M_x = 0 \; ; \; \Sigma M_y = 0 \; , \; \Sigma M_z = 0$$

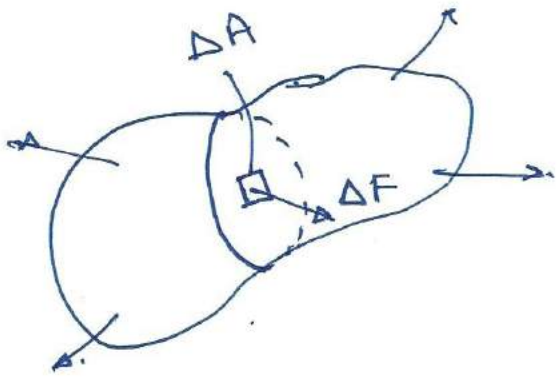
→ Internal Forces

$P_i$ → External forces




Distributed over entire area.



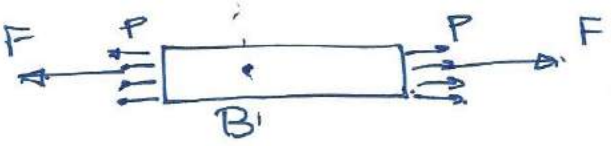

Avg. Force per unit area $= \frac{\Delta F}{\Delta A}$

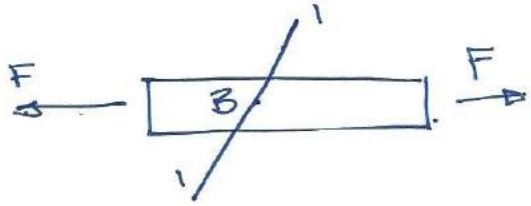
Stress at a point $p = \lim_{\Delta A \to 0} \frac{\Delta F}{\Delta A}$

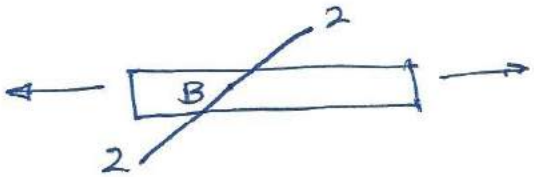
↳ acting on $\Delta A$

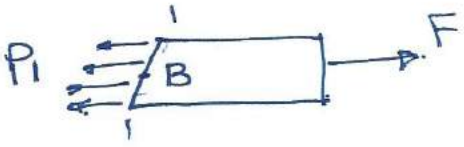
Is stress a vector.

Consider :-



$P > P_1 > P_2$

B → same point → multiple stresses.

stress at a point → wrong.

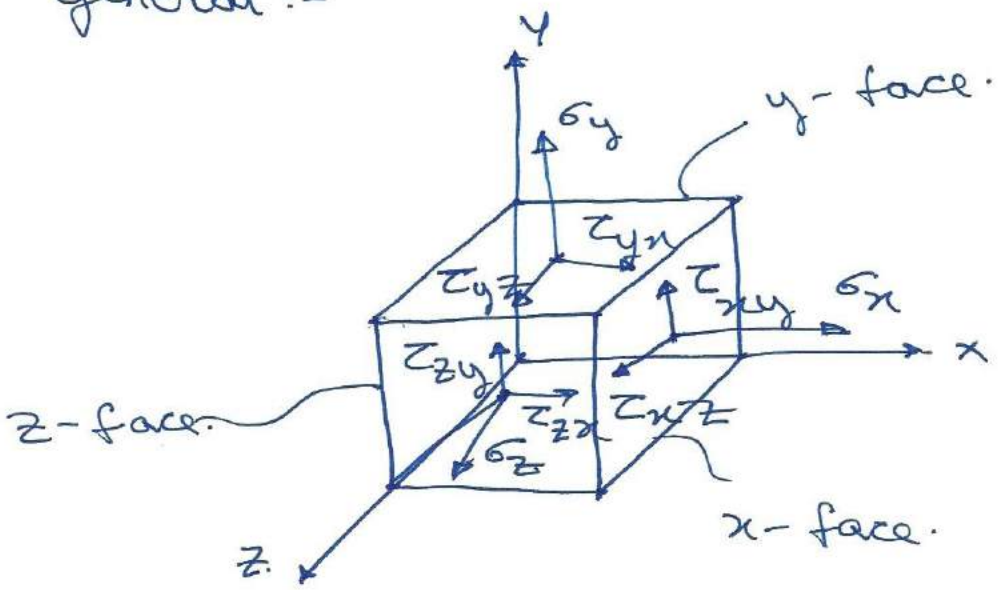
state of stress at a point

↳ also dependent on the plane.

∴ stress is not a vector.

↳ 2nd order tensor.

In general :-




Normal stresses.

$\sigma_x$

$\sigma_y$

$\sigma_z$

Shear stress

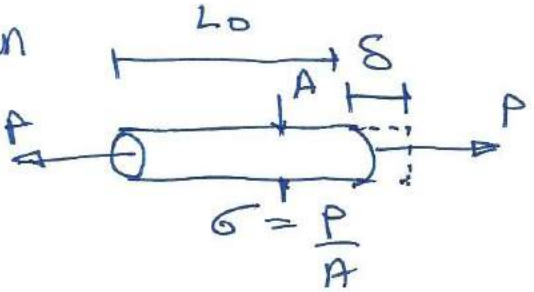
$\tau_{xy} = \tau_{yx}$

$\tau_{xz} = \tau_{zx}$

$\tau_{yz} = \tau_{zy}$

(3) Deformation : strain

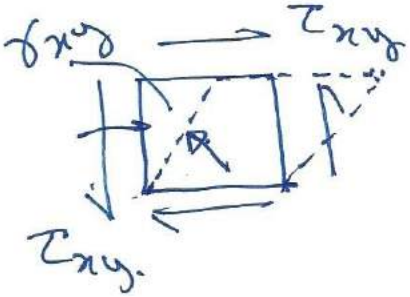
Normal strain :




$$\epsilon = \frac{\delta}{L_0}$$

Shear strain :-

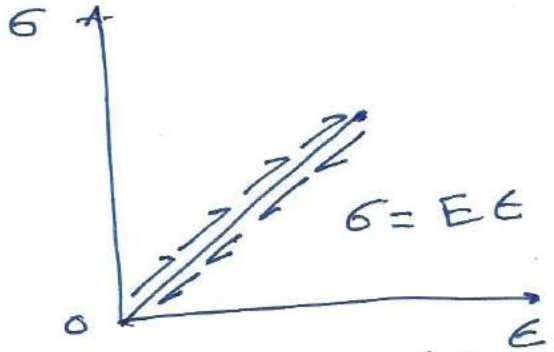
change of angle.



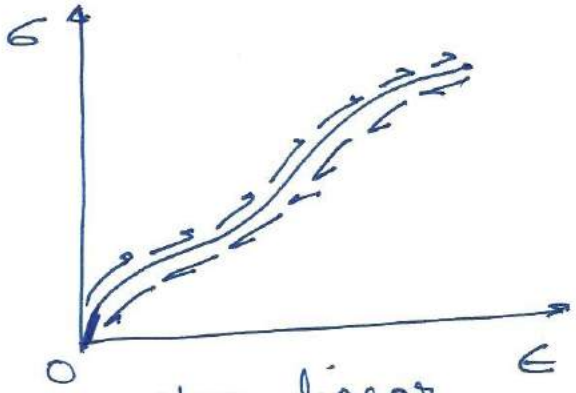
II } Constitutive Equations

↓

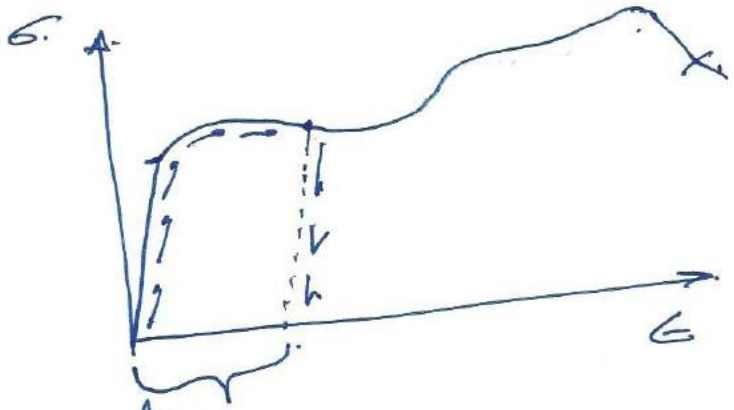
Materials properties.




Linear Elastic

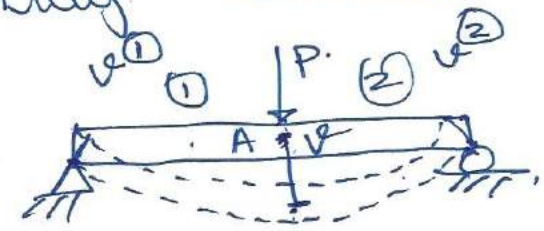



Non-linear Elastic




permanent set

III } Compatibility conditions




Displacement & slope should be "smooth"

i.e. continuous → $v_A^{(1)} = v_A^{(2)}$

$v_A'^{(1)} = v_A'^{(2)}$

function is compatible if single valued at every point.

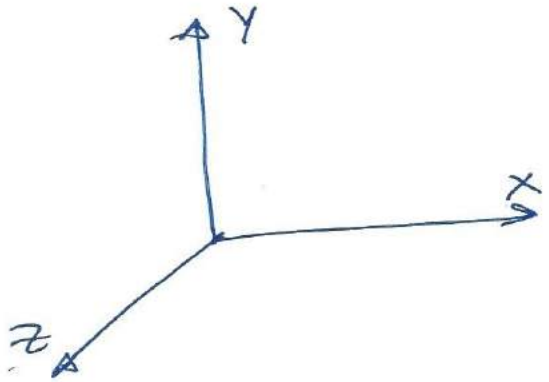
Chapter 1 :- cartesian Tensors.

$$\sigma_{xx} = \frac{E}{1-\nu^2}\left[\epsilon_{xx} + \nu\,\epsilon_{yy} + 0\,\epsilon_{xy}\right]$$

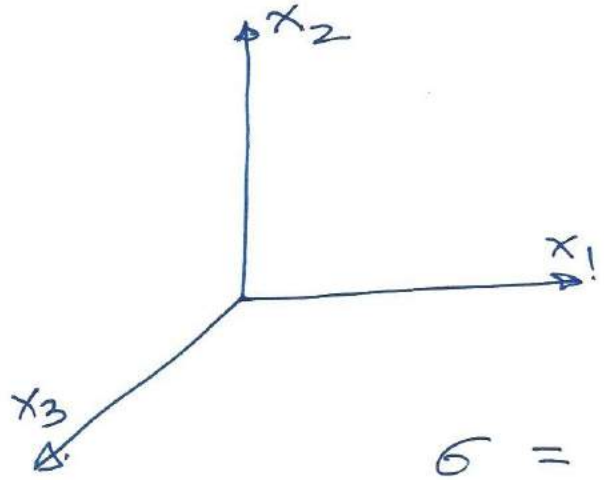
$$\sigma_{yy} = \frac{E}{1-\nu^2}\left[\epsilon_{yy} + \nu\,\epsilon_{xx} + 0\,\epsilon_{xy}\right]$$

$$\boxed{\sigma_{ij} = C_{ijkl}\,\epsilon_{ij}}$$

Consider state of stress at a point




$$\sigma = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix}$$




$$\sigma_{xx} \longleftrightarrow \sigma_{11}$$

$$\sigma_{xy} \longleftrightarrow \sigma_{12}$$

$$\sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$$

$\sigma_{ij}$ ; $i$ & $j$ = 1 to 3

indices

range.

$i=1,\ j=3 \rightarrow \sigma_{13}$

# - Notation & summation convention

★ Notation :- A set of n variables : $x_1, x_2, x_3 \cdots, x_n$ ; can be written simply as

$$x_i \; ; \; i = 1, 2, 3, \cdots, n$$

$i \rightarrow$ index
$x \rightarrow$ variable
$1 - n \rightarrow$ range of the index.

Consider the Eq$^n$ :- $a_1 x_1 + a_2 x_2 + a_3 x_3 = p$

$x_1, x_2, x_3 \rightarrow$ space variables.

$a_i, p \rightarrow$ constants.

This equation represents a plane in the 3-space $(x_1, x_2, x_3)$.

we can write this as :- $\sum_{i=1}^{3} a_i x_i = p$

Drop the summation sign ($\Sigma$) $\Rightarrow$ $a_i x_i = p$ (same)

- Summation convention :- The repetition of an index in a product term denotes a summation with respect to that index over the range of that index.

★ In this class ; range $\rightarrow$ 1 - 3.

Assume :- $\bar{A} = (A_1, A_2, A_3)$, $\bar{B} = (B_1, B_2, B_3)$

$A_i$ $B_i$, $i = 1, 2, 3$

$A_i B_j$ = $A_1B_1$, $A_1B_2$, $A_1B_3$ $i=1$, $j=1,2,3$

$i = 1, 2, 3$ $A_2B_1$, $A_2B_2$, $A_2B_3$ $i=2$

$j = 1, 2, 3$ $A_3B_1$, $A_3B_2$, $A_3B_3$ $i=3$

↳ 9 different terms

$A_i B_i$ = $A_1B_1 + A_2B_2 + A_3B_3$

dummy index ← same index

1 term.

$= A_j B_j = A_k B_k$

~~Dummy~~

Dummy index :- An index that is repeated and thus summed over.

Free index :- An index that is not repeated & thus not summed over.

Integral :- $\int_a^b f(x)dx = \int_a^b f(y)dy$

dummy variables.

Example :-

$A_i B_i = A_1B_1 + A_2B_2 + A_3B_3$

$A_{ii} = A_{11} + A_{22} + A_{33}$

↕ summation.

$$A_{ijj} = A_{i11} + A_{i22} + A_{i33} \quad -(1).$$

$j \rightarrow$ dummy.

$i \rightarrow$ free $\rightarrow 1,2,3.$

$i=1 \;:-\; A_{111} + A_{122} + A_{133}$

$i=2 \;:\; A_{211} + A_{222} + A_{233}$

$i=3 \;:\; A_{311} + A_{322} + A_{333}.$

Assignment :-

(1) Expand :- $b_{ij}\, b_{jk} = 0$ , $i, j, k = 1,2,3.$

(2) Expand :- $F_i = ma_i$ $\quad i = 1,2,3.$

Soln :-

(2) $F_i = ma_i \qquad i = 1,2,3.$

$\hookrightarrow$ $F_1 = ma_1$

$F_2 = ma_2$

$F_3 = ma_3$

(1) $b_{ij}\, b_{jk} = 0$

$j \rightarrow$ dummy index $\rightarrow$ sum over its range.

$\rightarrow b_{i1}\, b_{1k} + b_{i2}\, b_{2k} + b_{i3}\, b_{3k} = 0$

$i, k \rightarrow$ free indices.

$i = 1$ ; $k \to 1, 2, 3$ :-

$k = 1 : b_{11} b_{11} + b_{12} b_{21} + b_{13} b_{31} = 0$ — (1)

$k = 2 : b_{11} b_{12} + b_{12} b_{22} + b_{13} b_{32} = 0$ — (2)

$k = 3 : b_{11} b_{13} + b_{12} b_{23} + b_{13} b_{33} = 0$ — (3)

3 Eq$^{n}$ :- $i = 2$

3 Eq$^{n}$ :- $i = 3$

- Rules:-

(1) If a subscript occurs twice in a term in an equation, then it must be summed over its range. These are called repeated / dummy indices.

Eg: $C_{iKK} = C_{i11} + C_{i22} + C_{i33}$

(2) If a subscript occurs once in a term then it must ~~occur~~ occur once in every other term. These are the free indices.

Eg :- $F_i = m a_i$

(3) If a subscript occurs more than twice in a term, then it is a mistake.

Eg :- $A_{iij} B_{ij}$

— Contraction :- A special summing operation where two free indices are set equal to each other.

$A_{ijkl} \xrightarrow[\text{over } i]{\text{contraction}} A_{iikl} =$

$A_{ijkl}$ : $4^{th}$

$A_{iikl}$ : $i$ ↕ dummy ↓ summation ↓ $2^{nd}$.

when contraction is applied to an $n^{th}$ order tensor, we get a new tensor which is of the order $n-2$

- Summation convention :-

$$a_{ii} = a_{11} + a_{22} + a_{33}$$

$$a_{ij} b_{ij} = a_{11}b_{11} + a_{12}b_{12} + a_{13}b_{13} + a_{21}b_{21} + \cdots$$

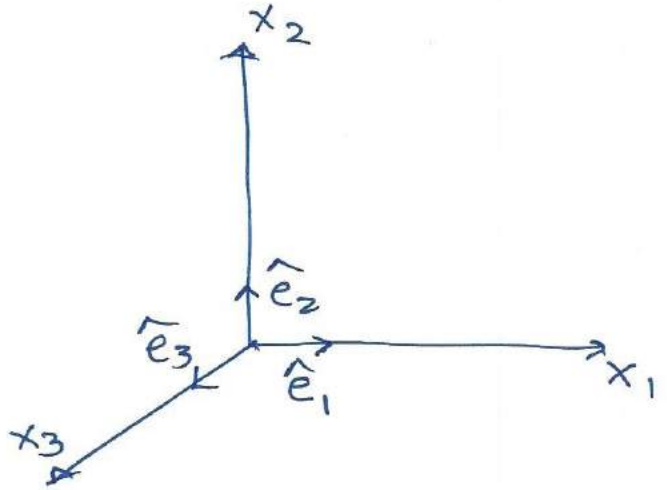
- Special symbols :-

- Base vectors : Unit vectors parallel to the coordinate axis.

$\hat{e}_1$ $\parallel^{el}$ to $x_1$-axis

$\hat{e}_2$ $\parallel^{el}$ to $x_2$

$\hat{e}_3$ $\parallel^{el}$ to $x_3$




Consider a vector

$$\hat{V} = V_1\hat{e}_1 + V_2\hat{e}_2 + V_3\hat{e}_3$$

$$\hat{V} = V_i\hat{e}_i$$

- Kronecker delta : $\delta_{ij}$ ; $i \& j = 1, 2, 3$

$$\delta_{ij} = 1 \quad \text{if } i = j$$

$$= 0 \quad \text{if } i \neq j$$

in matrix form: $[\delta_{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Properties of $\delta_{ij}$

(i) Symmetrical : $\delta_{ps} = \delta_{sp}$

(ii) Trace : $\delta_{pp} = \delta_{11} + \delta_{22} + \delta_{33} = 3$

(iii) Contraction :- $A_{ik}\delta_{ij} = A_{1k}\delta_{1j} + A_{2k}\delta_{2j} + A_{3k}\delta_{3j}$

$= A_{jk}$

Example :- Consider base vectors $\hat{e}_1, \hat{e}_2, \hat{e}_3$

$\hat{e}_1 \cdot \hat{e}_1 = 1$ , $\hat{e}_1 \cdot \hat{e}_2 = 0$

$\therefore \hat{e}_i \cdot \hat{e}_j = \delta_{ij}$

- Permutation symbol : $\epsilon_{ijk}$

$$[a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\det[a_{ij}] = a_{11}a_{22}a_{33} + a_{21}a_{32}a_{13} + a_{31}a_{12}a_{23} - a_{11}a_{32}a_{23} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}$$

$$|a_{ij}| = \epsilon_{rst}\, a_{r1}\, a_{s2}\, a_{t3}$$

$\epsilon_{rst} = 0$ if $r=s$, $s=t$, $r=t$

$= 1$ if $rst$ form an even permutation (clockwise)

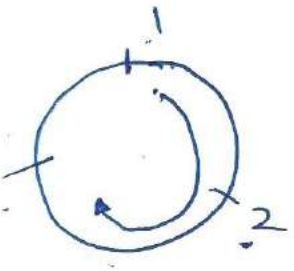
$= -1$ if $rst$ form an odd permutation

$\epsilon_{ijk}$

$\epsilon_{123} = 1$ , $\epsilon_{113} = 0$

$\epsilon_{231} = 1$ , $\epsilon_{212} = 0$

$\epsilon_{321} = -1$ , $\epsilon_{132} = -1$




Properties of $\epsilon_{ijk}$

- $\epsilon_{ijk}$ is anti-symmetric : $\epsilon_{ijk} = -\epsilon_{ikj}$
- Trace : $\epsilon_{iik} = \epsilon_{11k} + \epsilon_{22k} + \epsilon_{33k} = 0$
- Contraction : $\epsilon_{ijs}\epsilon_{pqs} = \delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}$

$\star$ $\hat{A} = A_1\hat{e}_1 + A_2\hat{e}_2 + A_3\hat{e}_3$ , $\hat{B} = B_1\hat{e}_1 + B_2\hat{e}_2 + B_3\hat{e}_3$

$$\hat{A}\times\hat{B} = \begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix} = (A_2B_3 - B_2A_3)\hat{e}_1 + (A_3B_1 - A_1B_3)\hat{e}_2 + (A_1B_2 - A_2B_1)\hat{e}_3$$

$$\hat{A}\times\hat{B} = (A_i\hat{e}_i)\times(B_j\hat{e}_j)$$

$$= A_iB_j(\hat{e}_i\times\hat{e}_j)$$

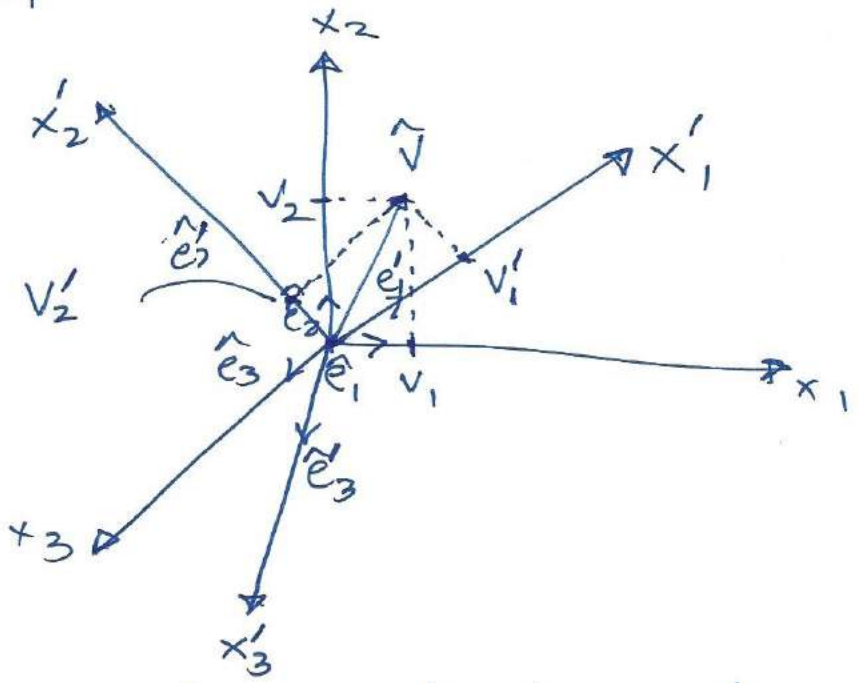
$\hat{e}_1\times\hat{e}_2 = \hat{e}_3$

In general : $\hat{e}_i\times\hat{e}_j = \epsilon_{ijk}\hat{e}_k$

$\therefore \hat{A}\times\hat{B} = A_iB_j\epsilon_{ijk}\hat{e}_k$

# 1-2 Coordinate Transformation

$$\hat{V} = V_1\hat{e}_1 + V_2\hat{e}_2 + V_3\hat{e}_3$$
$$= V_i\hat{e}_i$$




w.r.t. primed co-ordinate system

$$\hat{V} = V_1'\hat{e}_1' + V_2'\hat{e}_2' + V_3'\hat{e}_3'$$
$$= V_1\hat{e}_1 + V_2\hat{e}_2 + V_3\hat{e}_3$$

$$\therefore \hat{V} = V_i\hat{e}_i = V_j'\hat{e}_j'$$

---

$$\Rightarrow \quad V_j'\hat{e}_j' = V_j\hat{e}_j$$

$$V_j'\hat{e}_j'\cdot\hat{e}_i' = V_j\hat{e}_j\cdot\hat{e}_i'$$

$$V_j'\delta_{ij} = V_j\hat{e}_j\cdot\hat{e}_i'$$

contraction: $V_i' = V_j\hat{e}_j\cdot\hat{e}_i' = V_j\hat{e}_i'\cdot\hat{e}_j$

$$V_i' = e_i' \cdot \hat{e}_j \, V_j$$

Define: $B_{ij} = \hat{e}_i' \cdot \hat{e}_j$ → Direction cosine.

① —— $\therefore$ $\boxed{V_i' = B_{ij} V_j}$ Transformation rule.

dummy. ← from unprimed to primed.

We can show that :- $\boxed{V_i = B_{ji} V_j'}$ —— ②.

from primed to unprimed

Plug ② into ①

i.e. $V_i' = B_{ij} B_{kj} V_k'$

$= \delta_{ik} V_k'$

$\therefore B_{ij} B_{kj} = \delta_{ik}$ → orthogonality

In matrix form:-

$$\hat{V} = \{V_1, V_2, V_3\}, \quad [B_{ij}] = \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix}$$

$V_i' = B_{ij} V_j \Rightarrow \{V'\} = [B]\{V\}$

or $\{V\} = [B]^T \{V'\}$

orthogonal :- $\beta^{-1} = \beta^T$

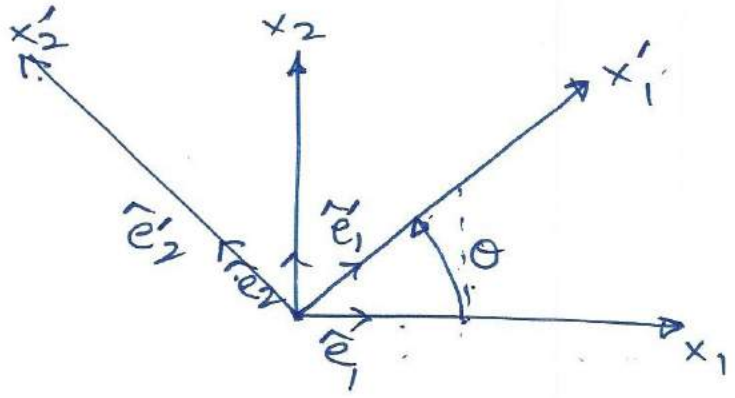
i.e. $[\beta][\beta^T] = [I]$

$\det[\beta] = \pm 1$

Eg :- In 2-D :-

$\hat{e}_1 = (1, 0)$

$\hat{e}_2 = (0, 1)$




$\hat{e}_1' = (\cos\theta, \sin\theta)$

$\hat{e}_2' = (-\sin\theta, \cos\theta)$

$\beta_{ij} = \hat{e}_i' \cdot \hat{e}_j$

$\therefore \beta_{11} = \hat{e}_1' \cdot \hat{e}_1 = \cos\theta$ ; $\beta_{12} = \hat{e}_1' \cdot \hat{e}_2 = \sin\theta$

$\beta_{21} = \hat{e}_2' \cdot \hat{e}_1 = -\sin\theta$ , $\beta_{22} = \cos\theta$

$$[\beta] = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

Given :- $\hat{v} = (v_1, v_2, v_3)$ , Find $\hat{v}'$

$v_i' = \beta_{ij} v_j = \beta_{i1} v_1 + \beta_{i2} v_2 + \beta_{i3} v_3$

($j$ : dummy)

$v_1' = \beta_{11} v_1 + \beta_{12} v_2 = \cos\theta\, v_1 + \sin\theta\, v_2$

$v_2' = \beta_{21} v_1 + \beta_{22} v_2 = -\sin\theta\, v_1 + \cos\theta\, v_2$

- Dr. Markus Linke, Hamburg university
  1:00 PM 2:00 PM — Tues JB 127
  - Wed CH 107.

## 1.3 Scalar, vector and Tensor fields.

Field quantity :- A quantity which is a function of position.

$$\phi = \phi(x_1, x_2, x_3, t)$$

Consider 2 fixed co-ordinate systems

$\{x_1, x_2, x_3\}$ ; $\{x_1', x_2', x_3'\}$

↓ unprimed ↓ primed.

$$x_i' = B_{ij} x_j$$

→. Consider $t_i$, has 3-components in $\{x_1, x_2, x_3\}$. If $t_i'$ in the primed system satisfies. $t_i' = t_i'\{x_1', x_2', x_3'\}$

$$t_i' = B_{ij} t_j \quad , \quad B_{ij} = \hat{e}_i' \cdot \hat{e}_j$$

then; $t_i$ is a 1ST order tensor field/vector

Consider a quantity with 9 components in $\{x_1, x_2, x_3\} \longrightarrow t_{ij}$

If $t_{ij}$ satisfies :→ $t'_{ij} = B_{im} B_{jn} t_{mn}$

then $t_{ij}$ is a second order tensor.

i.e. $[t'] = [B][t][B]^T$

In general :- $n^{th}$ order tensor.

- $3^n$ components
- $n$ $[B]$ matrices in transformation

i.e. $t'_{ijk\ldots} = B_{im} B_{jn} \ldots\ldots t_{mnp\ldots}$

or $t_{ijk\ldots} = B_{mi} B_{nj} \ldots\ldots t'_{mnp\ldots}$

- $0^{th}$ order tensor : independent of co-ordinate system.
  - → Scalar ; "invariant"

Eg : Temperature

→ Isotropic tensor :- A tensor whose components remain unchanged under a co-ordinate transformation.

$$t'_{ij} = t_{ij} = B_{ik} B_{jl} t_{kl}$$

Examples:-

| Order | Tensor. |
|---|---|
| 0 | 1 |
| 1 | — |
| 2 | $\delta_{ij}$ |
| 3 | $\epsilon_{ijk}$ |
| 4 | $\delta_{ij}\delta_{pq}$ ; $\delta_{ip}\delta_{jq}$ ; $\delta_{pj}\delta_{iq}$ |
| 5 | $\epsilon_{ijk}\delta_{pq}$ ; → all permutation |
| 6 | $(\delta_{ij}\delta_{pq}\delta_{rs})$ → 4 |

- Calculus of tensor fields.

$$V_i' = V_i'(x_1, x_2, x_3)$$

$$V_i' = V_i'(x_1', x_2', x_3')$$

Can we differentiate this tensor field?
↳ w.r.t to position.

Yes; if continuous.

→.
$$V_j' = B_{jk} V_k$$

$$\frac{\partial V_j'}{\partial x_i'} = B_{jk} \frac{\partial V_k}{\partial x_i'}$$

$$= B_{jk} \frac{\partial V_k}{\partial x_m} \frac{\partial x_m}{\partial x_i'} \quad \star.$$

$$x_m = \beta_{pm}\, x_p'$$

$$\therefore \frac{\partial x_m}{\partial x_i'} = \beta_{pm}\frac{\partial x_p'}{\partial x_i'} = \beta_{pm}\delta_{pi} = \beta_{im}$$

$$\frac{\partial V_j'}{\partial x_i'} = \beta_{jk}\frac{\partial V_k}{\partial x_m}\beta_{im} = \beta_{jk}\beta_{im}\frac{\partial V_k}{\partial x_m}$$

2 transformation matrices.

$\therefore$ 2nd order tensor.

$\Rightarrow$ A partial derivative of a tensor increases its order by 1.

$\rightarrow$ Consider a scalar field $\phi(x_1, x_2, x_3)$ then

$\frac{\partial \phi}{\partial x_i} \rightarrow$ is a 1st order tensor

Grad :- $[\nabla\phi]_i = \frac{\partial \phi}{\partial x_i}$ ; $\frac{\partial}{\partial x_i} = \partial_i$

Div :- $\nabla . F = \frac{\partial F_i}{\partial x_i}$ $i \rightarrow$ dummy.

$= \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \frac{\partial F_3}{\partial x_3}$

Curl :- $[\nabla \times F]_i = \epsilon_{ijk}\frac{\partial F_k}{\partial x_j}$

Given :- $t_{ijk}(x_1, x_2, x_3)$. then

$$\frac{\partial t_{ijk}}{\partial x_p} = C_{ijkp} \rightarrow 4^{th} \text{ order.}$$

→ Quotient rule :-

$$C_{ijk} D_k = A_{ij}$$

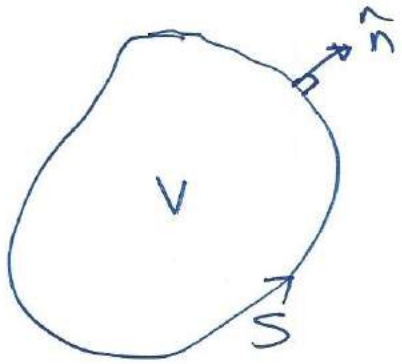
$C_{ijk}$ : Tensor, $A_{ij}$ : Tensor.

Then $D_k$ is also a tensor.

$$D_k' = B_{kl} D_l$$

— Integral Theorem :

Consider a tensor field $T_{jkm....}(x)$ in a volume V bounded by surface S.




Then :-

$$\int_V \frac{\partial}{\partial x_i} T_{jkm....} \, dV = \int_S n_i \cdot T_{jkm....} \, ds$$

$\hat{n}$ → unit normal.

Other forms :-

If we have a scalar field

$$\int_S n_i \phi \, ds = \int_V \frac{\partial \phi}{\partial x_i} \, dV$$

- If we have a vector field

$$\int_S n_i v_i \, ds = \int_V \frac{\partial v_i}{\partial x_i} dV \rightarrow \text{Gauss's Theorem.}$$

$$\int_S \epsilon_{ijk} n_j v_k \, ds = \int_V \epsilon_{ijk} \frac{\partial v_k}{\partial x_j} dV$$

1.4 → Properties of second order tensors.

$\sigma_{ij}$ ; $\epsilon_{ij}$

stress tensor ; strain tensor.

- A tensor $S_{ij}$ is symmetric, if

$$S_{ij} = S_{ji} \Rightarrow [S]^T = [S]$$

- A tensor $A_{ij}$ is anti-symmetric, if

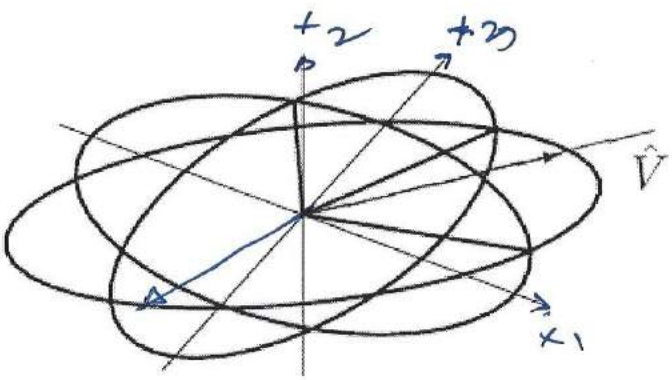
$$A_{ij} = -A_{ji} \Rightarrow [A]^T = -[A]$$

$A_{ij}$ → zero's along the diagonal.

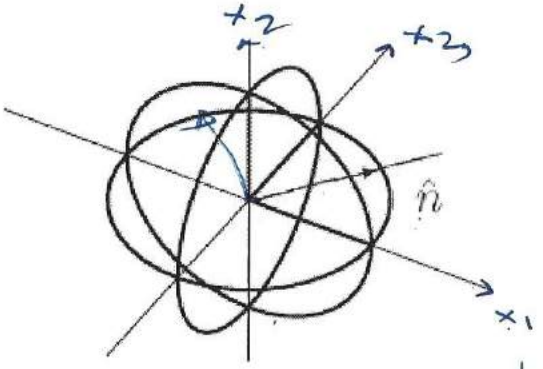
*. Any second order tensor, $T_{ij}$, can be written as the sum of a symmetric and anti-symmetric tensor.

$$T_{ij} = \underbrace{\frac{1}{2}(T_{ij} + T_{ji})}_{S_{ij}} + \underbrace{\frac{1}{2}(T_{ij} - T_{ji})}_{A_{ij}}$$

$$= S_{ij} + A_{ij}$$




$$V_i = T_{ij} n_j$$




$\hat{V}$ traces an ellipsoid

$\hat{n}$ traces a sphere.

$\therefore$ sphere traced by $\hat{n}$ is transformed into an ellipsoid traced by $\hat{V}$.

* In general, $\hat{V} \neq \hat{n}$ are not parallel.

Also :- $S_{ij} A_{ij} = 0$

i.e. contraction of a symmetric and anti-symmetric tensor is zero.

→ Principal values of symmetric tensors:

Let : $T_{ij}$ be a symmetric 2nd order tensor

$n_i$ be a unit vector.

Then, we can write :-

$$T_{ij} n_j = V_i$$

⇒ Contracting the unit vector with a 2nd order symmetric tensor leads to a vector $V_i$

i.e. $n_j$ transformed to $V_i$

in general ; $n_j \parallel^{el} V_i$ → not true.

→ Are there any vectors that have same direction before and after transformation?

→ Assume :- Yes.

then : $V_i = \lambda n_i = T_{ij} n_j$

i.e. $T_{ij} n_j - \lambda n_i = 0$

i.e. $T_{ij} n_j - \lambda \delta_{ij} n_j = 0$

$$[T_{ij} - \lambda \delta_{ij}] n_j = 0 \longrightarrow 3 \text{ Eq}^n.$$

$$\begin{bmatrix} T_{11}-\lambda & T_{12} & T_{13} \\ T_{21} & T_{22}-\lambda & T_{23} \\ T_{31} & T_{32} & T_{33}-\lambda \end{bmatrix} \begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

For non-trivial sol$^n$ :-

$$\begin{vmatrix} T_{11}-\lambda & T_{12} & T_{13} \\ T_{21} & T_{22}-\lambda & T_{23} \\ T_{31} & T_{32} & T_{33}-\lambda \end{vmatrix} = 0$$

Expansion gives us :-

$$\lambda^3 - I_1 \lambda^2 + I_2 \lambda - I_3 = 0 \quad \text{**}$$

$I_1, I_2, I_3$ are invariants.

$$I_1 = T_{ii}$$

$$I_2 = \frac{1}{2}\left[T_{ii} T_{jj} - T_{ij} T_{ji}\right]$$

$$I_3 = \det[T_{ij}]$$

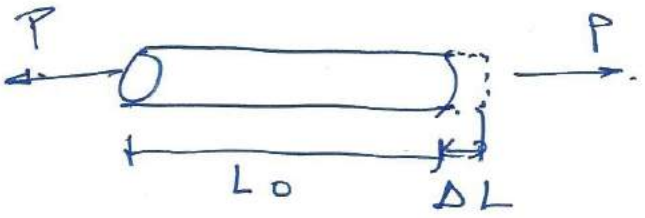
** has 3 sol$^n$ for $\lambda$.

$\lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)}$ (Eigen values).

$\hat{n}^1, \hat{n}^2, \hat{n}^3$ (eigen vector, principal direction).

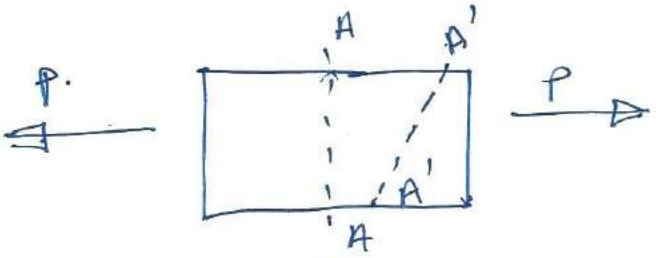
$\hat{n}^1 \perp \hat{n}^2 \perp \hat{n}^3$

# Chapter 2: Deformation

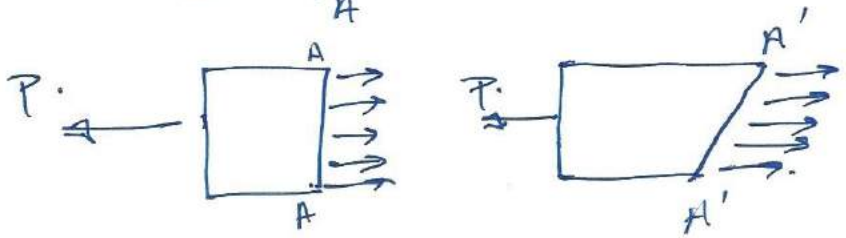



$$\epsilon = \frac{\Delta L}{L_0}$$

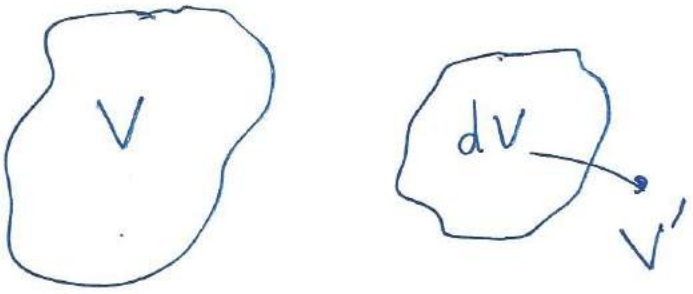
↳ Related to change in length.




$$\sigma = \frac{P}{A}$$




stress → related to areas.

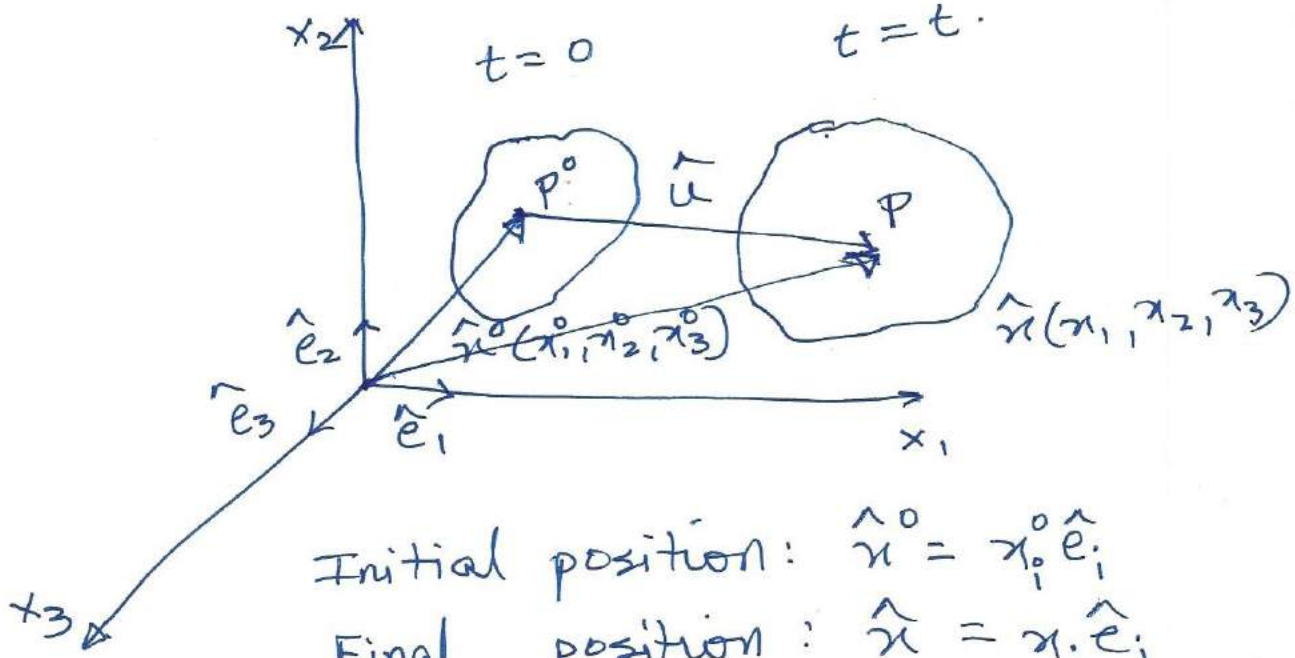



change in volume.
↓
mass.

Q] How does length, area & volume change under deformation?

→ Need a general discription of deformation.

## 2.1 General description of deformations.




Initial position: $\hat{x}^0 = x_i^0 \hat{e}_i$

Final position: $\hat{x} = x_i \hat{e}_i$

$\hat{u}$ : Displacement of $P^0$ to $P$

$x_i^0 \rightarrow$ undeformed configuration

$x_i \rightarrow$ deformed configuration

Motion:

$$x_i = x_i(x_i^0, t) \quad \text{or} \quad x_i^0 = x_i^0(x_i, t)$$

$$x_i = x_i(x_1^0, x_2^0, x_3^0, t) \qquad x_i^0 = x_i^0(x_1, x_2, x_3, t)$$

"Lagrangian" "Eulerian"

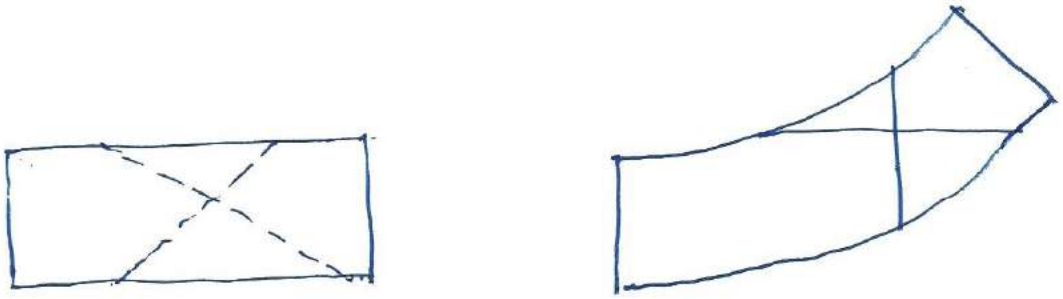
→ Lagrangian system: All quantities are expressed in terms of initial position coordinates and time.

undeformed. Deformed.

Lagrangian system.

Put a rectangular grid on undeformed body — determine what it will look like after deformation.

→ Eulerian system: All quantities expressed in terms of position coordinates at the time of interest.



undeformed. Deformed.

Eulerian system.

Put a rectangular grid on the deformed body — determine what it looked like in the original state.

Lagrangian → material description

Eulerian → spatial description

Consider $\phi(x_1^\circ, x_2^\circ, x_3^\circ, t)$

$$\frac{\partial \phi}{\partial t} = \frac{d\phi}{dt} \leftarrow \text{Local velocity}$$

In Eulerian system: $\overline{\phi}(x_1, x_2, x_3, t) \equiv \phi(x_1^\circ, x_2^\circ, x_3^\circ, t)$

$$\frac{d\overline{\phi}}{dt} = \frac{\partial \overline{\phi}}{\partial t} + \frac{\partial \overline{\phi}}{\partial x_j}\left(\frac{\partial x_j}{\partial t}\right) = \frac{\partial \overline{\phi}}{\partial t} + v_j \frac{\partial \overline{\phi}}{\partial x_j}$$

↑ convective velocity

DEFORMATION: Comparison of two states — intermediate states are not important. Time usually doesn't matter.

$$x_i = x_i(x_1^\circ, x_2^\circ, x_3^\circ), \quad x_i^\circ = x_i^\circ(x_1, x_2, x_3)$$

DISPLACEMENT: shortest distance traveled when a particle moves from one location to another.

$$\hat{u} = \hat{r} - \hat{r}_0 = x_i \hat{e}_i - x_i^\circ \hat{e}_i$$

$$u_i = x_i - x_i^\circ$$

Example:- Consider the motion

$$\left.\begin{aligned} x_1 &= \tfrac{1}{2}(x_1^\circ + x_2^\circ)e^{t} + \tfrac{1}{2}(x_1^\circ - x_2^\circ)e^{-t} \\ x_2 &= \tfrac{1}{2}(x_1^\circ + x_2^\circ)e^{t} - \tfrac{1}{2}(x_1^\circ - x_2^\circ)e^{-t} \end{aligned}\right\} (1)$$

Final state → in terms of initial state.

∴ Lagrangian.

Displacement vector:

$$u_i = x_i - x_i^\circ \quad : \quad \begin{aligned} u_1 &= x_1 - x_1^\circ \\ u_2 &= x_2 - x_2^\circ \end{aligned}$$

$$\left.\begin{aligned} u_1 &= \tfrac{1}{2}(x_1^\circ + x_2^\circ)e^{t} + \tfrac{1}{2}(x_1^\circ - x_2^\circ)e^{-t} - x_1^\circ \\ u_2 &= \tfrac{1}{2}(x_1^\circ + x_2^\circ)e^{t} - \tfrac{1}{2}(x_1^\circ - x_2^\circ)e^{-t} - x_2^\circ \end{aligned}\right\} (3)$$

$\hat{u}$ in terms of initial position.

We can invert (1) and write

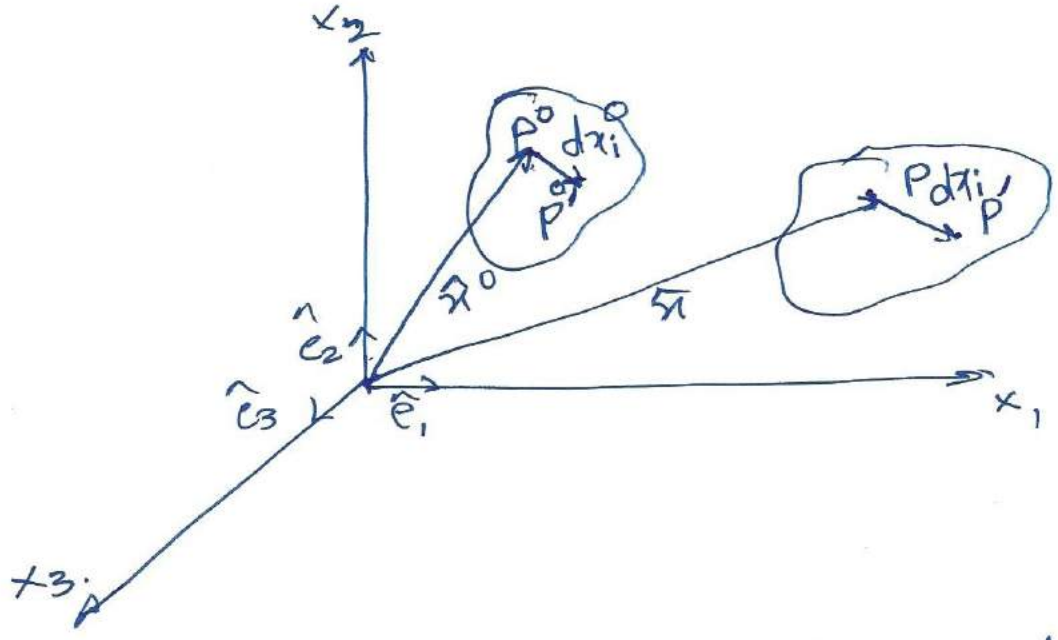
$$\left.\begin{aligned} x_1^\circ &= \tfrac{1}{2}(x_1 + x_2)e^{-t} + \tfrac{1}{2}(x_1 - x_2)e^{t} \\ x_2^\circ &= \tfrac{1}{2}(x_1 + x_2)e^{-t} - \tfrac{1}{2}(x_1 - x_2)e^{t} \end{aligned}\right\} (2)'$$

Then: $u_i = x_i - x_i^\circ$ :

$$\left.\begin{aligned} u_1 &= x_1 - \left[\tfrac{1}{2}(x_1 + x_2)e^{-t} + \tfrac{1}{2}(x_1 - x_2)e^{t}\right] \\ u_2 &= x_2 - \left[\tfrac{1}{2}(x_1 + x_2)e^{-t} - \tfrac{1}{2}(x_1 - x_2)e^{t}\right]. \end{aligned}\right\} (4)$$

(3) → (4). equivalent.

# DEFORMATION GRADIENT




Consider two point P and P' seperated by $dx_i^0$ in undeformed configuration. After deformation, they are seperated by $dx_i$

Point P :- $x_i = x_i(x_i^0)$

$P'$ :- $x_i + dx_i = x_i(x_i^0 + dx_i^0)$

$$\cong x_i(x_i^0) + \frac{\partial x_i}{\partial x_j^0} dx_j^0 + \frac{1}{2} \frac{\partial^2 x_i}{\partial x_j \partial x_k} dx_j^0 dx_k^0 + \cdots$$

↳ Taylor series.

If $dx_i^0$ is small ; i.e. P and P' are very close we can ignore higher order terms.

$$x_i' + dx_i = x_i' + \frac{\partial x_i}{\partial x_j^0} dx_j^0$$

$$dx_i = \frac{\partial x_i}{\partial x_j^0} dx_j^0$$

seperation in deformed state ↵ ↳ seperation in initial state.

$$\begin{Bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{Bmatrix} = \begin{bmatrix} \frac{\partial x_1}{\partial x_1^0} & \frac{\partial x_1}{\partial x_2^0} & \frac{\partial x_1}{\partial x_3^0} \\ \frac{\partial x_2}{\partial x_1^0} & \frac{\partial x_2}{\partial x_2^0} & \frac{\partial x_2}{\partial x_3^0} \\ \frac{\partial x_3}{\partial x_1^0} & \frac{\partial x_3}{\partial x_2^0} & \frac{\partial x_3}{\partial x_3^0} \end{bmatrix} \begin{Bmatrix} dx_1^0 \\ dx_2^0 \\ dx_3^0 \end{Bmatrix}$$

If the body is continuous, then the inverse should hold.

$$\begin{Bmatrix} dx_1^0 \\ dx_2^0 \\ dx_3^0 \end{Bmatrix} = \begin{bmatrix} \frac{\partial x_1^0}{\partial x_1} & \frac{\partial x_1^0}{\partial x_2} & \frac{\partial x_1^0}{\partial x_3} \\ \frac{\partial x_2^0}{\partial x_1} & \frac{\partial x_2^0}{\partial x_2} & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix} \begin{Bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{Bmatrix}$$

i.e. $$dx_i^0 = \frac{\partial x_i^0}{\partial x_j} dx_j$$

$\frac{\partial x_i}{\partial x_j^0}$, $\frac{\partial x_i^0}{\partial x_j}$ are called deformation gradients.

They relate behavior of neighboring particles.

If $dx_i = \frac{\partial x_i}{\partial x_j^0} dx_j^0$

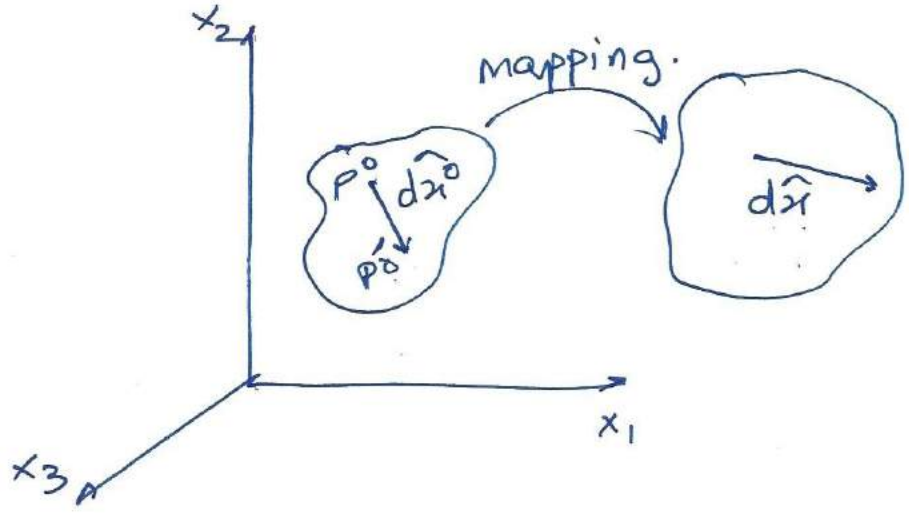
& deformation is continuous

$$\{dx_i^0\} = \left[\frac{\partial x_i}{\partial x_j^0}\right]^{-1} \{dx_j\}$$

This is true iff: $\det\left[\frac{\partial x_i}{\partial x_j^0}\right] \neq 0$

Jacobian: $J^0 = \det\left[\frac{\partial x_i}{\partial x_j^0}\right] = \epsilon_{ijk} \frac{\partial x_1}{\partial x_i^0} \frac{\partial x_2}{\partial x_j^0} \frac{\partial x_3}{\partial x_k^0}$

$$J = \det\left[\frac{\partial x_i^0}{\partial x_j}\right] = \epsilon_{ijk} \frac{\partial x_1^0}{\partial x_i} \frac{\partial x_2^0}{\partial x_j} \frac{\partial x_3^0}{\partial x_k}$$




$$dx_i = \frac{\partial x_i}{\partial x_j^0} dx_j^0 \quad ; \quad dx_i^0 = \frac{\partial x_i^0}{\partial x_j} dx_j$$

$\frac{\partial x_i}{\partial x_j^0}$ ; $\frac{\partial x_i^0}{\partial x_j}$ → Deformation gradients.

↳ Relates behavior of neighboring points.

If: $dx_i = \frac{\partial x_i}{\partial x_j^0} dx_j^0$

deformation is continuous.

$$\{dx_i^0\} = \left[\frac{\partial x_i}{\partial x_j^0}\right]^{-1} \{dx_j\}$$

This is satisfied if $\det\left[\frac{\partial x_i}{\partial x_j^0}\right] \neq 0$

Jacobian: $J^0 \equiv \det\left[\frac{\partial x_i}{\partial x_j^0}\right] = \epsilon_{ijk} \frac{\partial x_1}{\partial x_i^0} \frac{\partial x_2}{\partial x_j^0} \frac{\partial x_3}{\partial x_k^0}$

$$J = \det\left[\frac{\partial x_i^0}{\partial x_j}\right] = \epsilon_{ijk} \frac{\partial x_1^0}{\partial x_i} \frac{\partial x_2^0}{\partial x_j} \frac{\partial x_3^0}{\partial x_k}$$

★ $0 < J < \infty$ ; $0 < J^0 < \infty$.

★ important for the deformation to be physically possible.

- Jacobian plays an important role in restricting the path between the initial and final states.

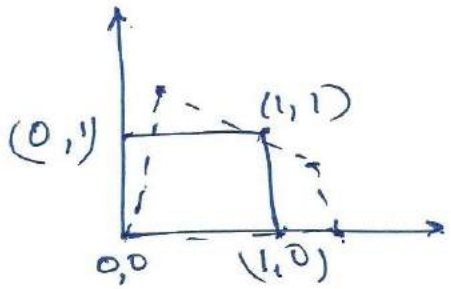
Example: Given deformation

$$x_1 = x_1^0 + K x_2^0 \qquad K \text{ is a parameter}$$

$$x_2 = 2x_1^0 + 3x_2^0$$

$$x_3 = x_3^0$$

→ The deformation gradient is




$$\left[\frac{\partial x_i}{\partial x_j^0}\right] = \begin{bmatrix} 1 & K & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Jacobian : $\det\left[\frac{\partial x_i}{\partial x_j^0}\right] = 3 - 2K$

$0 < J^0 < \infty$ $\qquad 3 - 2K = 0$

i.e. $K = 1.5$

i.e. $\underline{K < 1.5}$.

if $K = 2$

i.e. $x_1 = x_1^\circ + 2x_2^\circ$

$x_2 = 2x_1^\circ + 3x_2^\circ$

$x_3 = x_3^\circ$

~~if~~ $\boxed{K < 1.5.}$

Plot initial state $\longrightarrow$ Plot final state.

as you vary $K: 0 \longrightarrow 2$.

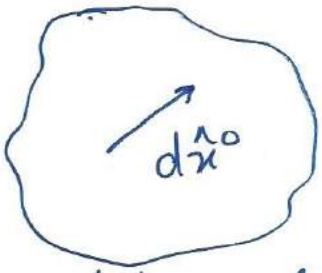
at $K = 1.5. \rightarrow$ Volume $= 0$.

↓

Not possible.

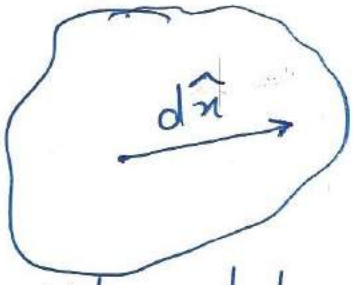
## 2.2. Deformation of lines, areas, and volumes.

LINES :-




Before deformation

$d\hat{x}^\circ = dx_i^\circ \hat{e}_i$



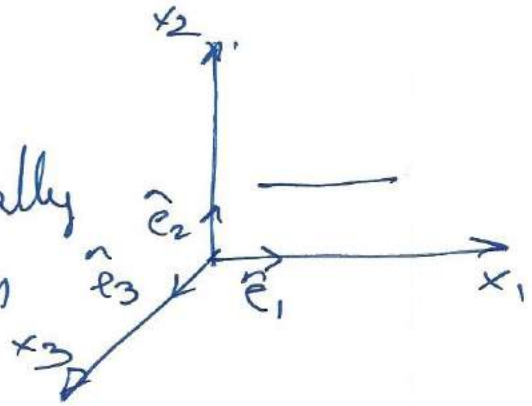

After deformation

$d\hat{x} = dx_i \hat{e}_i$

$$d\hat{x} = \frac{\partial x_i}{\partial x_j^\circ} dx_j^\circ \hat{e}_i$$

Example :-

Assume a line, $d\hat{x}^\circ$, originally oriented along $e_1$-direction



$$d\hat{x}^0 = dx_1^0 \hat{e}_1 + dx_2^0 \hat{e}_2 + dx_3^0 \hat{e}_3$$

$$d\hat{x}^0 = dx_1^0 \hat{e}_1$$

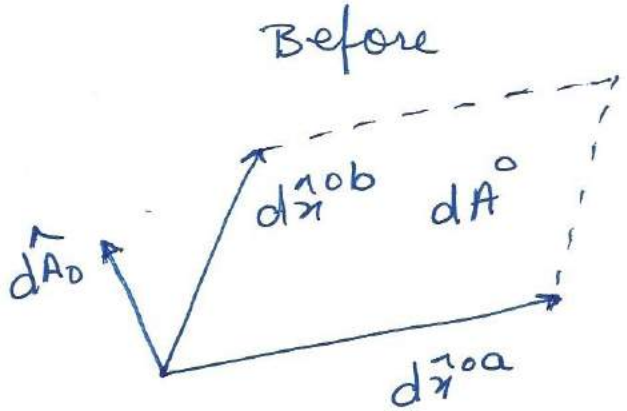
After deformation: $d\hat{x} = \frac{\partial x_i}{\partial x_j^0} dx_j^0 \hat{e}_i$

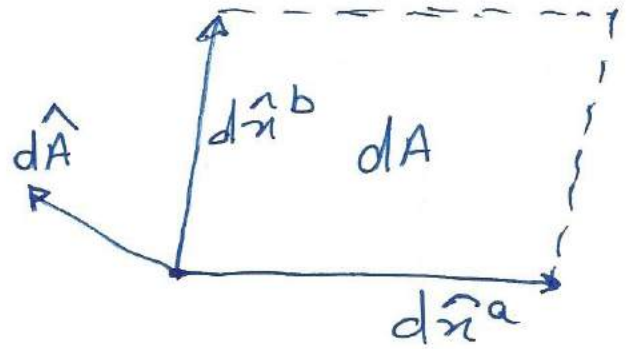
$$d\hat{x} = dx_1 \hat{e}_1 + dx_2 \hat{e}_2 + dx_3 \hat{e}_3$$

$$= \frac{\partial x_1}{\partial x_1^0} dx_1^0 \hat{e}_1 + \frac{\partial x_2}{\partial x_1^0} dx_1^0 \hat{e}_2 + \frac{\partial x_3}{\partial x_1^0} dx_1^0 \hat{e}_3$$

$\therefore$ $d\hat{x}^0$ is horizontal, but $d\hat{x}$ has all three components.

- AREAS




$$d\hat{A}^0 = d\hat{x}^{0a} \times d\hat{x}^{0b} \qquad d\hat{A} = d\hat{x}^a \times d\hat{x}^b$$

Area is a vector. The direction is given by a normal to the surface -

i.e. $dA_k^0 \hat{e}_k = dx_i^{0a} \hat{e}_i \times dx_j^{0b} \hat{e}_j = \epsilon_{ijk} dx_i^{0a} dx_j^{0b} \hat{e}_k$

in component notation:

$$dA_K^\circ = \epsilon_{ijK}\, dx_i^{\circ a}\, dx_j^{\circ b}$$

Deformation of areas; after some trivial math;

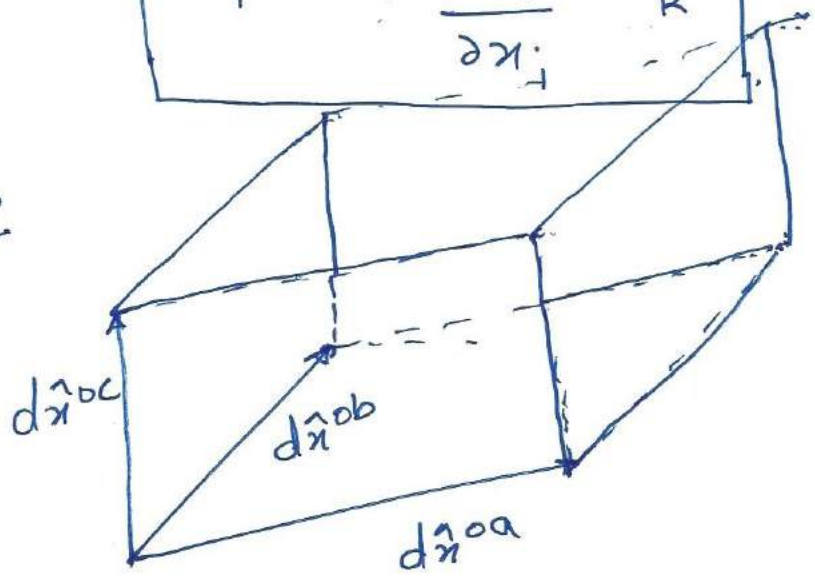
$$dA_K \frac{\partial x_K}{\partial x_i^\circ} = J^\circ dA_i^\circ$$

OR

$$\boxed{dA_i = J^\circ \frac{\partial x_K^\circ}{\partial x_i}\, dA_K^\circ}$$

$$J^\circ = \det\left[\frac{\partial x_i}{\partial x_j^\circ}\right]$$

## VOLUMES




Vol': $dV^\circ = \left(d\hat{x}^{\circ a} \times d\hat{x}^{\circ b}\right) \cdot d\hat{x}^{\circ c}$

Before: $dV^\circ = \epsilon_{pqr}\, dx_p^{\circ a}\, dx_q^{\circ b}\, dx_r^{\circ c}$

After: $dV = \epsilon_{ijK}\, dx_i^a\, dx_j^b\, dx_K^c$

Deformation of volumes:

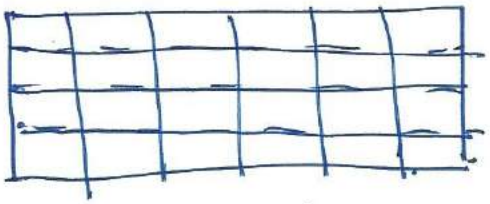
$$\boxed{dV = J^\circ dV^\circ}$$

$$J^\circ = \det\left[\frac{\partial x_i}{\partial x_j^\circ}\right]$$

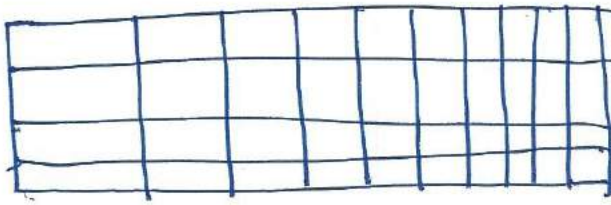
Conservation of mass : mass = density × volume

i.e. $\rho^0 dV^0 = \rho dV = \rho J^0 dV^0$

i.e. $\rho^0 = \rho J^0$ i.e. $J^0 = \frac{\rho^0}{\rho}$ ; $J = \frac{\rho}{\rho^0}$



original (homogeneous).



inhomogeneous.

Show that :-

$$\frac{\partial}{\partial x_i}(J^0) = \frac{\partial}{\partial x_i}\left(\frac{\rho^0}{\rho}\right) = 0$$

$$\frac{\partial}{\partial x_i^0}(J) = \frac{\partial}{\partial x_i^0}\left(\frac{\rho}{\rho^0}\right) = 0.$$

Example:- Assume the deformation

$$x_1 = 2x_1^0 + x_2^0 \quad , \quad x_2 = x_1^0 + x_2^0 \quad , \quad x_3 = x_3^0$$

How is a line initially in the $\hat{e}_1$ direction deformed?

$\rightarrow$ Deformation gradient :-

$$\left[\frac{\partial x_i}{\partial x_j^0}\right] = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} ; \text{ i.e. } I = 1$$

$\therefore$ Deformation is possible.

Let initial length be $ds_0$. We can write the line as

$$d\hat{x}^0 = dx_i^0 = ds_0\{1, 0, 0\}$$

After deformation :

$$d\hat{x} = \frac{\partial x_i}{\partial x_j^0} dx_j^0 \hat{e}_i$$

$$= \left(\frac{\partial x_1}{\partial x_1^0} dx_1^0 + \frac{\partial x_1}{\partial x_2^0} dx_2^0 + \frac{\partial x_1}{\partial x_3^0} dx_3^0\right)\hat{e}_1$$

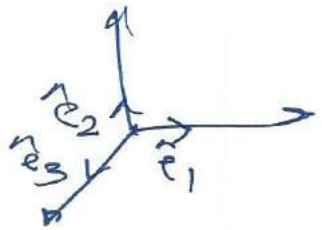
$$+ (\quad)\hat{e}_2$$

$$+ (\quad)\hat{e}_3$$

$$d\hat{x} = 2ds_0\hat{e}_1 + ds_0\hat{e}_2 + 0\hat{e}_3$$

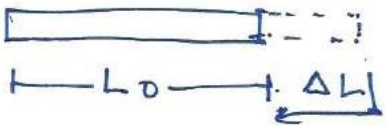
New length : $ds = |d\hat{x}| = \sqrt{(2ds_0)^2 + (ds_0)^2} = \sqrt{5}\, ds_0$

Direction :- $\frac{d\vec{x}}{|d\vec{x}|} = \frac{2ds_0\hat{e}_1 + ds_0\hat{e}_2}{\sqrt{5}\,ds_0}$




## 2.3 STRAIN

Elementary strain measures




- Engineering strain : $e = \frac{\text{change in length}}{\text{original length}} = \frac{\Delta L}{L_0}$

- True strain : $e^T = \frac{\text{change in length}}{\text{Final length}} = \frac{\Delta L}{L_0 + \Delta L}$

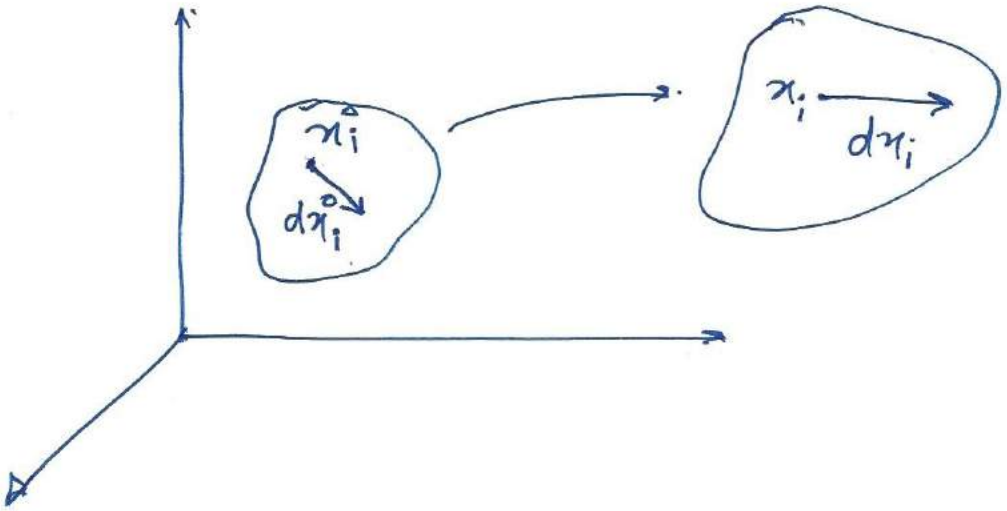
- Logarithmic strain : $e^N = \int_{L_0}^{L} \text{true strain}$

$$= \int_{L_0}^{L} \frac{dl}{l} = \ln\left(\frac{L}{L_0}\right)$$

Relationship :-

$$e^T = \frac{e}{1+e} \quad ; \quad e^N = \ln(1+e)$$

# Tensorial strain measure




Initial distance : $ds_0^2 = dx_i^0 dx_i^0 = (dx_1^0)^2 + (dx_2^0)^2 + (dx_3^0)^2$

Final distance : $ds^2 = dx_i dx_i = dx_1^2 + dx_2^2 + dx_3^2$

We have : $$dx_i = \frac{\partial x_i}{\partial x_j^0} dx_j^0$$

i.e. $$ds^2 = dx_i dx_i$$

$$ds^2 = \frac{\partial x_m}{\partial x_i^0} \frac{\partial x_m}{\partial x_j^0} dx_i^0 dx_j^0$$

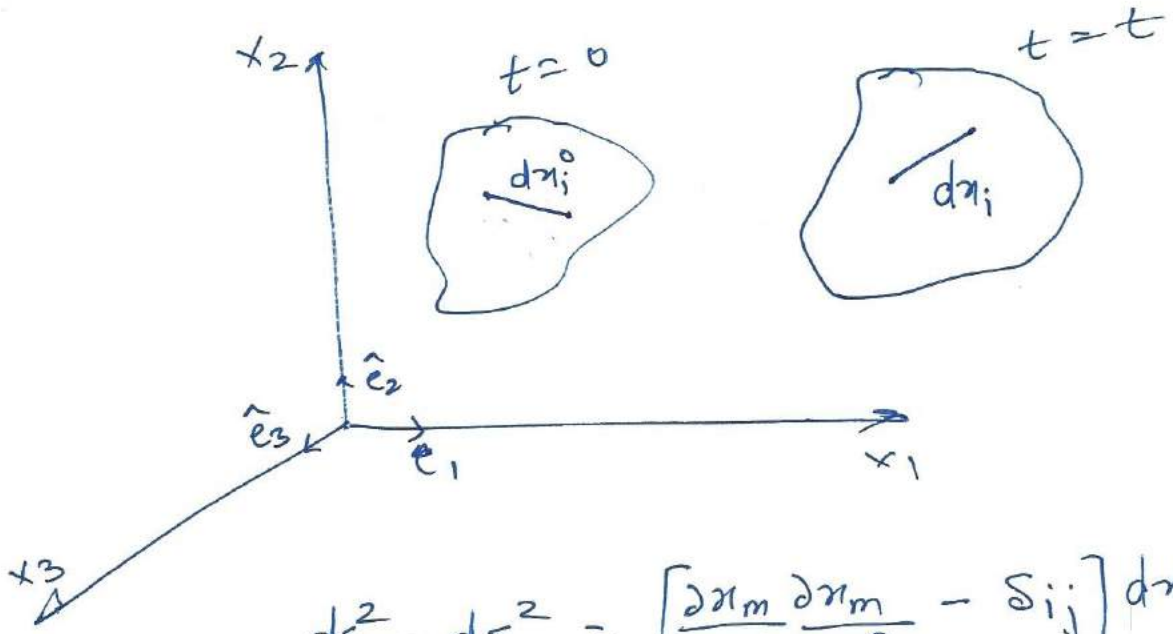
if $ds = ds_0$ : $ds^2 - ds_0^2 = 0$

i.e. $$ds^2 - ds_0^2 = ds^2 - dx_i^0 dx_i^0$$

$$ds^2 - ds_0^2 = \left[ \frac{\partial x_m}{\partial x_i^0} \frac{\partial x_m}{\partial x_j^0} - \delta_{ij} \right] dx_i^0 dx_j^0$$

↳ measure of relative displacements
i.e. insensitive to rotation.

Lecture 7.




$$ds^2 - ds_0^2 = \left[\frac{\partial x_m}{\partial x_i^\circ}\frac{\partial x_m}{\partial x_j^\circ} - \delta_{ij}\right] dx_i^\circ dx_j^\circ$$

↳ Lagrangian description.

$$= 2E_{ij}\, dx_i^\circ dx_j^\circ$$

Using Eulerian variables, we can write

$$ds^2 - ds_0^2 = \left[\delta_{ij} - \frac{\partial x_m^\circ}{\partial x_i}\frac{\partial x_m^\circ}{\partial x_j}\right] dx_i dx_j$$

$$= 2e_{ij}\, dx_i dx_j$$

i.e. $ds^2 - ds_0^2 = 2E_{ij}\, dx_i^\circ dx_j^\circ = 2e_{ij}\, dx_i dx_j$

where:

$E_{ij} = \frac{1}{2}\left[\frac{\partial x_m}{\partial x_i^\circ}\frac{\partial x_m}{\partial x_j^\circ} - \delta_{ij}\right]$ → Lagrangian strain tensor.

$e_{ij} = \frac{1}{2}\left[\delta_{ij} - \frac{\partial x_m^\circ}{\partial x_i}\frac{\partial x_m^\circ}{\partial x_j}\right]$ → Eulerian strain tensor.

$E_{ij}$, $e_{ij}$ are symmetric 2nd order tensors.

Proof: $ds^2 - ds_0^2$ must be the same in any co-ordinate system.

* i.e. $ds^2 - ds_0^2 = 2E_{ij}\, dx_i^0 dx_j^0 = 2E'_{ij}\, dx_i'^0 dx_j'^0$

$dx_i^0$ is a 1st order tensor.

i.e. $dx_i'^0 = \beta_{ij}\, dx_i^0$ i.e. $dx_i^0 = \beta_{ji}\, dx_i'^0$

i.e. * $\rightarrow ds^2 - ds_0^2 = 2E_{ij}\, \beta_{pi}\, \beta_{qj}\, dx_p'^0 dx_q'^0 = 2E'_{ij}\, dx_i'^0 dx_j'^0$

$\Rightarrow$ $E'_{ij} = \beta_{ip}\, \beta_{jq}\, E_{pq}$ $\rightarrow$ Q.E.P.

$\hookrightarrow$ Transformation law for 2nd order tensors.

In matrix form:-

$$2[E_{ij}] = \left[\frac{\partial x_m}{\partial x_i^0}\right]\left[\frac{\partial x_m}{\partial x_j^0}\right]^T - [I]$$

$$2[e_{ij}] = [I] - \left[\frac{\partial x_m^0}{\partial x_i}\right]\left[\frac{\partial x_m^0}{\partial x_j}\right]^T$$

$$\boxed{E_{ij} = e_{pq}\frac{\partial x_p}{\partial x_i^0}\frac{\partial x_q}{\partial x_j^0}}$$

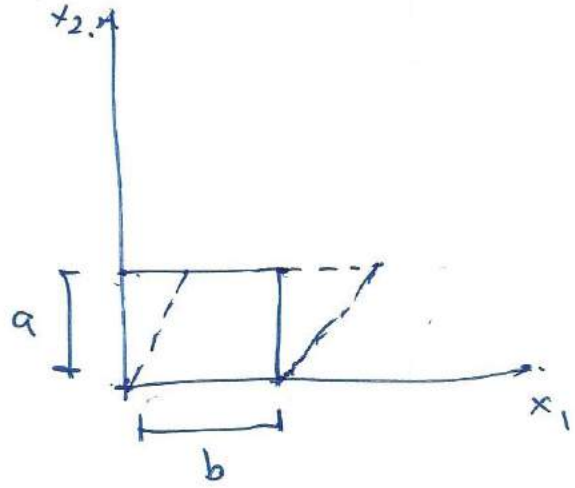
Example :- Consider the 2D deformation

$$x_1 = x_1^0 + k x_2^0$$

$$x_2 = x_2^0$$

→ Find $E_{ij}$

$$E_{ij} = \frac{1}{2}\left[\frac{\partial x_m}{\partial x_i^0}\frac{\partial x_m}{\partial x_j^0} - \delta_{ij}\right] \qquad i,j,m \to 1 \text{ to } 2.$$




$$x_1 = x_1^0 + k x_2^0$$

$$x_2 = x_2^0$$

$$\frac{\partial x_1}{\partial x_1^0} = 1 + 0 = 1$$

$$\frac{\partial x_2}{\partial x_1^0} = 0$$

$$E_{11} = \frac{1}{2}\left[\frac{\partial x_m}{\partial x_1^0}\frac{\partial x_m}{\partial x_1^0} - \delta_{11}\right]$$

$$= \frac{1}{2}\left[\frac{\partial x_1}{\partial x_1^0}\frac{\partial x_1}{\partial x_1^0} + \frac{\partial x_2}{\partial x_1^0}\frac{\partial x_2}{\partial x_1^0} - 1\right]$$

$$= \frac{1}{2}\left[1 + 0 - 1\right] = 0$$

$$E_{12} = \frac{1}{2}\left[\frac{\partial x_m}{\partial x_1^0}\frac{\partial x_m}{\partial x_2^0} - \delta_{12}\right]$$

$$= \frac{1}{2}\left[k + 0 - 0\right] = \frac{k}{2}$$

$$E_{21} = E_{12} = \frac{k}{2}$$

$$E_{22} = \frac{k^2}{2}$$

$$E_{ij} = \begin{bmatrix} 0 & k/2 \\ k/2 & k^2/2 \end{bmatrix}$$

→. Principal strains and invariants.

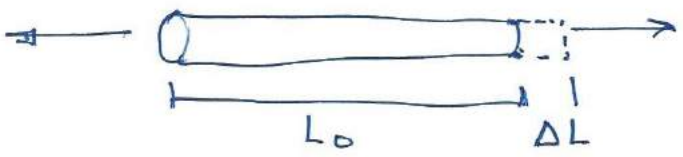
Pg 1.25 → $T_{ij} \rightarrow E_{ij}$

→ $E_{ij} n_j^\circ - \lambda n_i^\circ = 0$

or $[E_{ij} - \lambda \delta_{ij}] n_j^\circ = 0$

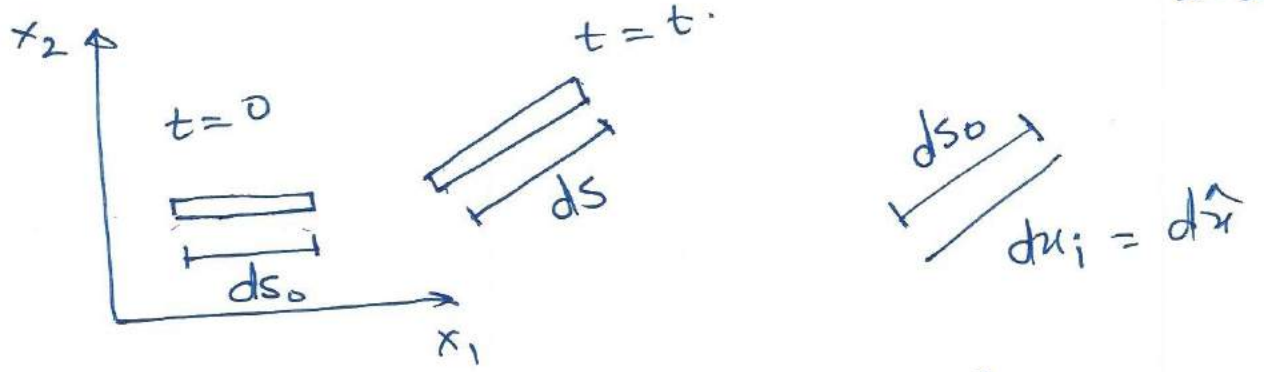
Principal strain : $\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$ , $n_j^\circ$

- Physical interpretation of normal strains




$E_1 = \frac{\Delta L}{L_0}$ = Normal strain.

"Extension"




Before deformation : $d\hat{x}^0 = \{ds_0, 0, 0\}$

Extension $E_1 = \dfrac{ds - ds_0}{ds_0}$

↳ How does normal strain relate $E_{ij}$

$$ds^2 - ds_0^2 = 2E_{ij}\, dx_i^0\, dx_j^0$$

Here, we have : $dx_1^0 = ds_0$, $dx_2^0 = dx_3^0 = 0$

$$ds^2 - ds_0^2 = 2E_{11}\, dx_1^0\, dx_1^0 = 2E_{11} ds_0^2$$

$$ds^2 - ds_0^2 = 2E_{11} ds_0^2$$

$$E_1 = \frac{ds - ds_0}{ds_0} \Rightarrow ds = (1 + E_1)\, ds_0$$

$$ds^2 - ds_0^2 = 2E_{11} ds_0^2$$

i.e. $(1+E_1)^2 ds_0^2 - ds_0^2 = 2E_{11} ds_0^2$

i.e. $E_{11} = E_1 + \dfrac{E_1^2}{2}$ or $E_1 = \sqrt{(1+2E_{11})} - 1$

$E_{22} = E_2 + \dfrac{E_2^2}{2}$ or $E_2 = \sqrt{(1+2E_{22})} - 1$

$E_{33} = E_3 + \dfrac{E_3^2}{2}$ or $E_3 = \sqrt{(1+2E_{22})} - 1$

$E_{11}, E_{22}, E_{33}$ → Normal components of strain.

Consider :- $$E_1 = \frac{ds - ds_0}{ds_0}$$

If we elongate the element → $E_1$ can increase without bounds ⟶ $E_{11}$ increases without bounds.

If we shrink the element ⟶ $\Delta L = -L_0$ i.e. line shrinks to zero. Cannot shrink anymore. $ds = 0$ ; $E_1 = -1$

$$-1 < E_1 < \infty \quad ; \quad -0.5 < E_{11} < \infty$$

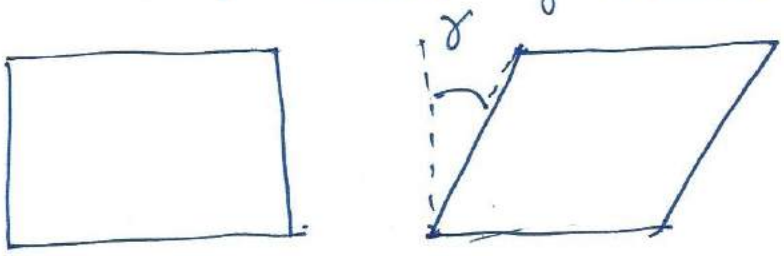
Apply binomial expansion to $E_1$

$$E_1 = \left(1 + E_{11} - \frac{1}{2} E_{11}^2 + \cdots\right) - 1$$

$$E_1 = E_{11} - \frac{1}{2} E_{11}^2 + \cdots$$

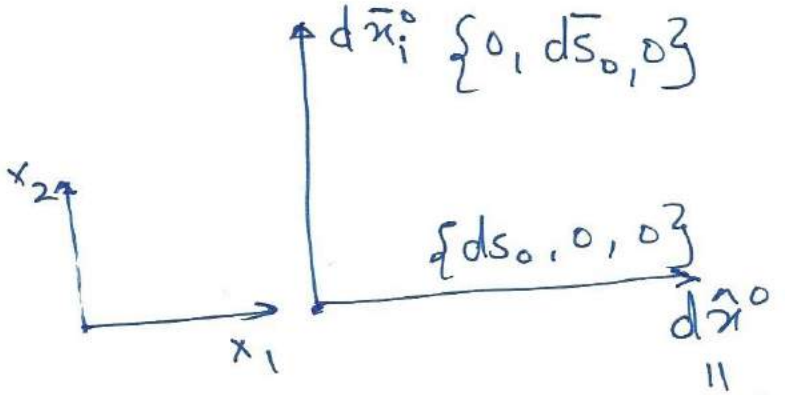
If $E_{11} << 1$, then $E_1 \simeq E_{11}$ for small deformation.

- Physical interpretation of shear strains



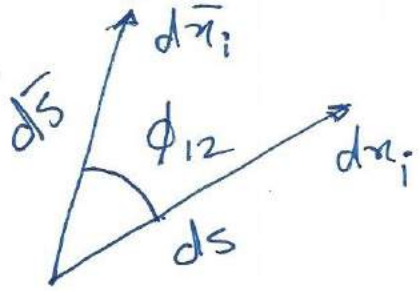

$\gamma \rightarrow$ shear strain.

change in angle.




$d\hat{x}^0 = dx_i^0$

Before deformation




$|d\hat{x}| = ds$

$|d\hat{\bar{x}}| = d\bar{s}$

i.e- $dx_1^0 = ds_0$ , $dx_2^0 = dx_3^0 = 0$

$d\bar{x}_2^0 = d\bar{s}_0$ , $d\bar{x}_1^0 = d\bar{x}_3^0 = 0$

$$d\hat{x} \cdot d\hat{\bar{x}} = ds\, d\bar{s} \cos\phi_{12} = dx_i\, d\bar{x}_i$$

$$dx_i\, d\bar{x}_i = \frac{\partial x_i}{\partial x_m^0} \frac{\partial x_i}{\partial x_m^0} dx_k^0\, d\bar{x}_m^0$$

$$= \frac{\partial x_i}{\partial x_1^0} \frac{\partial x_i}{\partial x_2^0} dx_1^0\, d\bar{x}_2^0 \quad \text{(all others are zero)}.$$

$$= 2 E_{12}\, dx_1^0\, d\bar{x}_2^0$$

$$ds\, d\bar{s} \cos\phi_{12} = 2E_{12} dx_1^0\, d\bar{x}_2^0$$

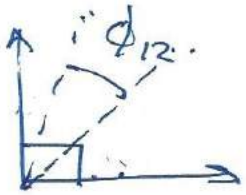
$$= 2E_{12}\, ds_0\, d\bar{s}_0$$

We had :- $ds = (1+E_1)ds_0$ ; $d\bar{s} = (1+E_2)d\bar{s}_0$

$$\therefore \quad \cos\phi_{12} = \frac{2E_{12}}{(1+E_1)(1+E_2)}$$

Change in angle (measure of shear) = Initial angle − Final angle.

$$\alpha_{12} = \frac{\pi}{2} - \phi_{12}$$




Rearranging :-

$$\sin\alpha_{12} = \frac{2E_{12}}{\sqrt{1+2E_{11}}\sqrt{1+2E_{22}}}$$

$E_{11}$, $E_{22}$, $E_{12}$ all contribute to shear.

But $\alpha_{12} = 0$ if $E_{12} = 0$

Thus, $E_{12}$ controls shear deformation.

$$-1 < \sin\alpha_{12} < 1 \quad \text{i.e.} \quad -\frac{\pi}{2} < \alpha_{12} < \frac{\pi}{2}$$

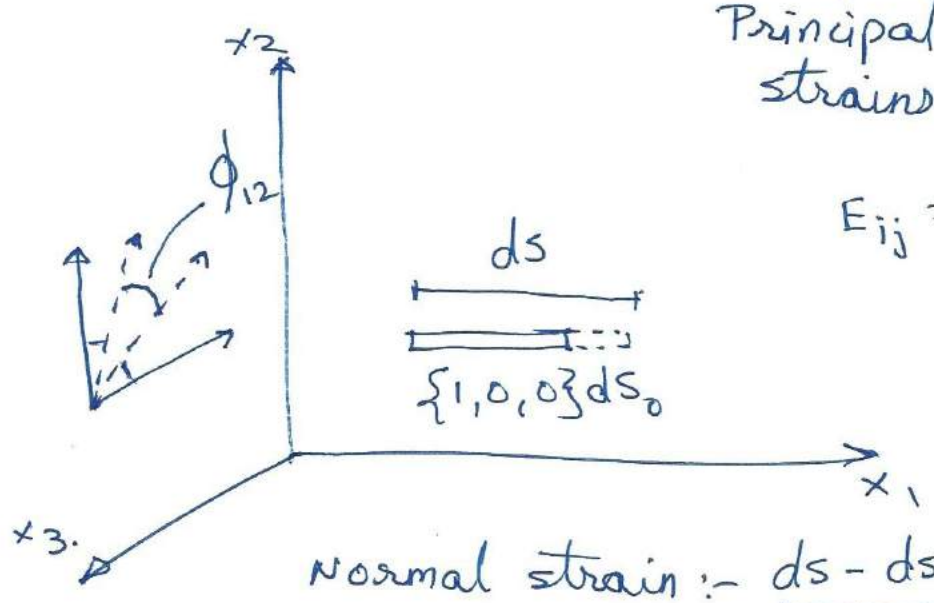
$$-\sqrt{1+2E_{11}}\sqrt{1+2E_{22}} < 2E_{12} < \sqrt{1+2E_{11}}\sqrt{1+2E_{22}}$$

Lagrangian strain Tensor : $E_{ij} = \frac{1}{2}\left[\frac{\partial x_m}{\partial x_i^0}\frac{\partial x_m}{\partial x_j^0} - \delta_{ij}\right]$.

- 2nd order tensor
- Symmetric : $E_{ij} = E_{ji}$
- Principal strains : $\left[E_{ij} - \lambda \delta_{ij}\right]^T n_j^0 = 0$

$\lambda$ → Principal strains; $n_j^0$ → Principal directions

$$E_{ij} = \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{21} & E_{22} & E_{23} \\ E_{31} & E_{32} & E_{33} \end{bmatrix}$$




Normal strain :- $\frac{ds - ds_0}{ds_0} = E_1$

$E_1 = \sqrt{1+2E_{11}} - 1$

$E_{11}$ & $E_1$ are related.

$E_{11}$ → a measure of normal strain

- Change of angle of two lines initially $\perp^{ar}$ to each other : $\alpha_{12} = \frac{\pi}{2} - \phi_{12}$

$$\sin\alpha_{12} = \frac{2E_{12}}{\sqrt{1+2E_{11}}\sqrt{1+2E_{22}}} \rightarrow E_{12} = 0 \leftrightarrow \alpha_{12} = 0$$

↓ measure of shear strain.

Also:- $-1 < E_1 < \infty$ ; $-0.5 < E_{11} < \infty$

For small deformation: $E_{11} \simeq E_1$

$$-\frac{\pi}{2} < \alpha_{12} < \frac{\pi}{2} \Rightarrow -\sqrt{1+2E_{11}}\sqrt{1+2E_{22}} < 2E_{12}$$

$$2E_{12} < \sqrt{1+2E_{11}}\sqrt{1+2E_{22}}$$

- Physical interpretation of Eulerian strains.

$$e_1 = \frac{ds - ds_0}{ds}$$

$$e_1 = 1 - \sqrt{1-2e_{11}}$$

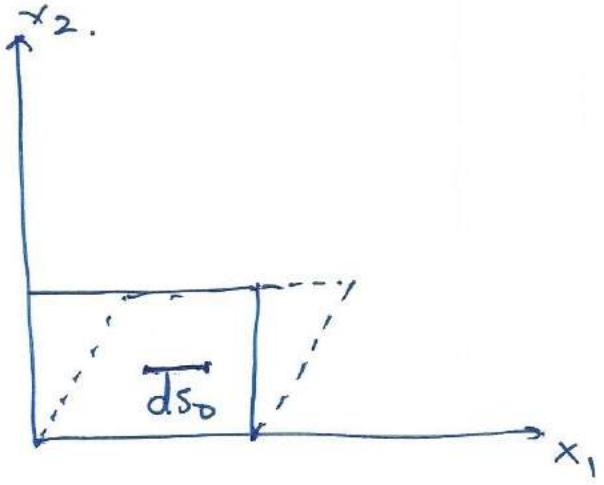
$$\sin \beta_{12} = \frac{2e_{12}}{\sqrt{1-2e_{11}}\sqrt{1-2e_{22}}} \quad , \beta_{12} = \phi_{12} - \frac{\pi}{2}$$

$$-\infty < e_1 < 1 \quad ; -\infty < e_{11} < 0.5$$

Example:- $x_1 = x_1^0 + Kx$

$x_2 = x_2^0$

$$E_{ij} = \begin{bmatrix} 0 & K/2 \\ K/2 & K^2/2 \end{bmatrix}$$




→. Element $\parallel^{el}$ to $x_1$- axis.

Find the extension $\left(\frac{ds}{ds_0}\right)$

$d\hat{x}_0 = \{ds_0, 0\}$

$\downarrow \quad \downarrow$

$dx_1^0 \quad dx_2^0$

We have :-

$$ds^2 - ds_0^2 = 2E_{ij}\, dx_i^0\, dx_j^0$$

$$E_{ij} = \begin{bmatrix} 0 & K/2 \\ K/2 & K^2/2 \end{bmatrix} \quad , \quad dx_1^0 = ds_0 \quad , \quad dx_2^0 = 0$$

$i,j \rightarrow 1 \& 2$

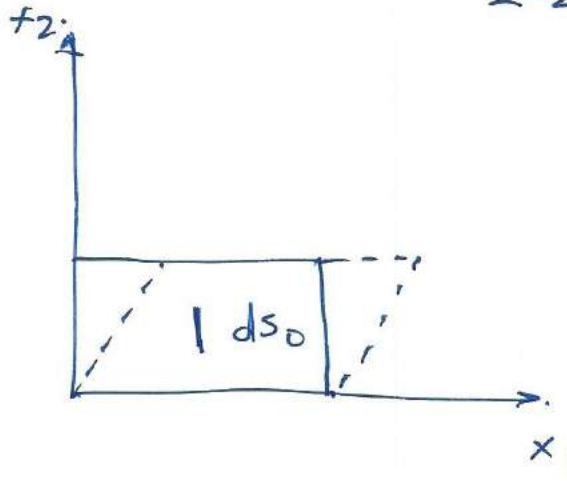
$$ds^2 - ds_0^2 = 2E_{11}\, dx_1^0\, dx_1^0 + 2E_{12}\, dx_1^0\, dx_2^0 + 2E_{21}\, dx_2^0\, dx_1^0 + 2E_{22}\, dx_2^0\, dx_2^0$$

($dx_1^0$ → 0, $dx_2^0$ → 0)

$$ds^2 - ds_0^2 = 2E_{11}\, dx_1^0\, dx_1^0$$

$$ds^2 - ds_0^2 = 0 \quad \rightarrow \text{. No deformation.}$$

- consider element parallel to $x_2$- axis.




$$d\hat{x}_0 = \{0, ds_0\}$$
$$\downarrow \quad \downarrow$$
$$dx_1^0 \quad dx_2^0$$

$$ds^2 - ds_0^2 = 2E_{ij}\, dx_i^0\, dx_j^0$$

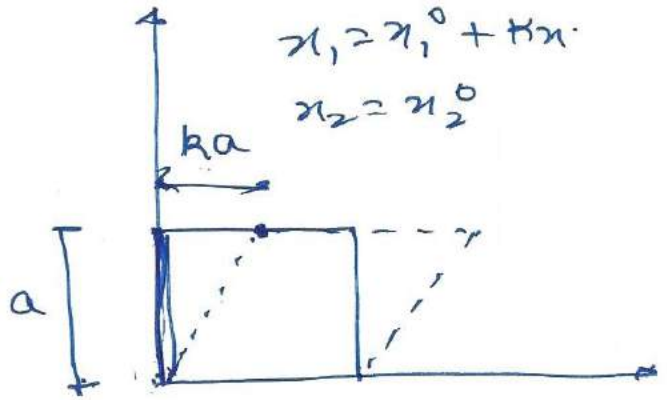
$$ds^2 - ds_0^2 = 2E_{22}\, dx_2^0\, dx_2^0 \qquad (dx_2^0 = ds_0)$$

$$ds^2 - ds_0^2 = 2E_{22}\, ds_0^2$$

i.e. $$ds = \sqrt{1 + k^2}\, ds_0$$

i.e. $$\frac{ds}{ds_0} = \sqrt{1+k^2} \quad \text{(Extension)}$$

Graphical method.




Let's say $ds_0 = a$

$$\epsilon = \frac{ds - ds_0}{ds_0}$$

$$ds^2 = ds_0^2 + (ka)^2$$

↳ Pythagoras.

$$= a^2 + k^2 a^2$$

$$\therefore ds = a\sqrt{1+k^2}$$

$$\therefore \frac{ds}{ds_0} = \sqrt{1+k^2} \longrightarrow \text{same as above.}$$

2-4 : Seen additional notes.

2-5 :- Deformation and strain in terms of displacements.

We had :-

$$u_M = x_m - x_m^\circ$$

i.e. $x_m = u_m + x_m^\circ$ ; $x_m^\circ = x_m - u_m$

$$\frac{\partial x_m}{\partial x_i^\circ} = \frac{\partial u_m}{\partial x_i^\circ} + \delta_{im} \quad ; \quad \frac{\partial x_m^\circ}{\partial x_i} = \delta_{im} - \frac{\partial u_m}{\partial x_i}$$

$\frac{\partial u_m}{\partial x_i^\circ}$ , $\frac{\partial u_m}{\partial x_i}$ $\longrightarrow$ Displacement gradients.

- strain and rotations in terms of disp.

$$E_{ij} = \frac{1}{2}\left[\frac{\partial x_m}{\partial x_i^\circ}\frac{\partial x_m}{\partial x_j^\circ} - \delta_{ij}\right]$$

$$= \frac{1}{2}\left[\left(\frac{\partial u_m}{\partial x_i^\circ} + \delta_{im}\right)\left(\frac{\partial u_m}{\partial x_j^\circ} + \delta_{jm}\right) - \delta_{ij}\right]$$

$$E_{ij} = \frac{1}{2}\left[\frac{\partial u_i}{\partial x_j^\circ} + \frac{\partial u_j}{\partial x_i^\circ} + \frac{\partial u_m}{\partial x_i^\circ}\frac{\partial u_m}{\partial x_j^\circ}\right]$$

$$e_{ij} = \frac{1}{2}\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{\partial u_m}{\partial x_i}\frac{\partial u_m}{\partial x_j}\right]$$

~~$E_{11} = \frac{1}{2}\left[\quad\right]$~~

$$E_{11} = \frac{\partial u_1}{\partial x_1^0} + \frac{1}{2}\left[\left(\frac{\partial u_1}{\partial x_1^0}\right)^2 + \left(\frac{\partial u_2}{\partial x_1^0}\right)^2 + \left(\frac{\partial u_3}{\partial x_1^0}\right)^2\right]$$

$$E_{12} = \frac{1}{2}\left[\frac{\partial u_1}{\partial x_2^0} + \frac{\partial u_2}{\partial x_1^0}\right] + \frac{1}{2}\left[\frac{\partial u_1}{\partial x_1^0}\frac{\partial u_1}{\partial x_2^0} + \frac{\partial u_2}{\partial x_1^0}\frac{\partial u_2}{\partial x_2^0} + \frac{\partial u_3}{\partial x_1^0}\frac{\partial u_3}{\partial x_2^0}\right]$$

* Non-linear terms make geometrical interpretation difficult.

A typical rotation can be written as

$$\tan\bar{\phi}_3 = \frac{\frac{1}{2}\left(\frac{\partial u_2}{\partial x_1^0} - \frac{\partial u_1}{\partial x_2^0}\right)}{\sqrt{\left(1 + \frac{\partial u_1}{\partial x_1^0}\right)\left(1 + \frac{\partial u_2}{\partial x_2^0}\right) - \frac{1}{4}\left(\frac{\partial u_1}{\partial x_2^0} + \frac{\partial u_2}{\partial x_1^0}\right)^2}}$$

$$w_{ij} = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i^0} - \frac{\partial u_i}{\partial x_j^0}\right) \rightarrow \text{Rotation tensor.}$$

$w_{ij} = 0 \rightarrow \bar{\phi} = 0$

$w_{ij} = -w_{ji}$ i.e. anti symmetric.

- Alternative expression for strains.

"Any 2nd order tensor can be written in terms of a symmetric and anti-symmetric tensor".

In Lagrangian system:

$$\bar{\epsilon}_{ij} = \frac{1}{2}\left[\frac{\partial u_j}{\partial x_i^\circ} + \frac{\partial u_i}{\partial x_j^\circ}\right], \quad \bar{\omega}_{ij} = \frac{1}{2}\left[\frac{\partial u_j}{\partial x_i^\circ} - \frac{\partial u_i}{\partial x_j^\circ}\right]$$

(symmetric) (anti-sym)

We can show:-

$$E_{ij} = \bar{\epsilon}_{ij} + \frac{1}{2}\left(\bar{\epsilon}_{kj} - \bar{\omega}_{kj}\right)\left(\bar{\epsilon}_{ki} - \bar{\omega}_{ki}\right)$$

↓ Linear. ↓ Non-linear → Rotation.

If :- $E_{ij} = 0$ → Rigid body motion.

$\bar{\epsilon}_{ij} = 0 \Rightarrow$ does not imply $E_{ij} = 0$ if $\bar{\omega}_{ij} \neq 0$

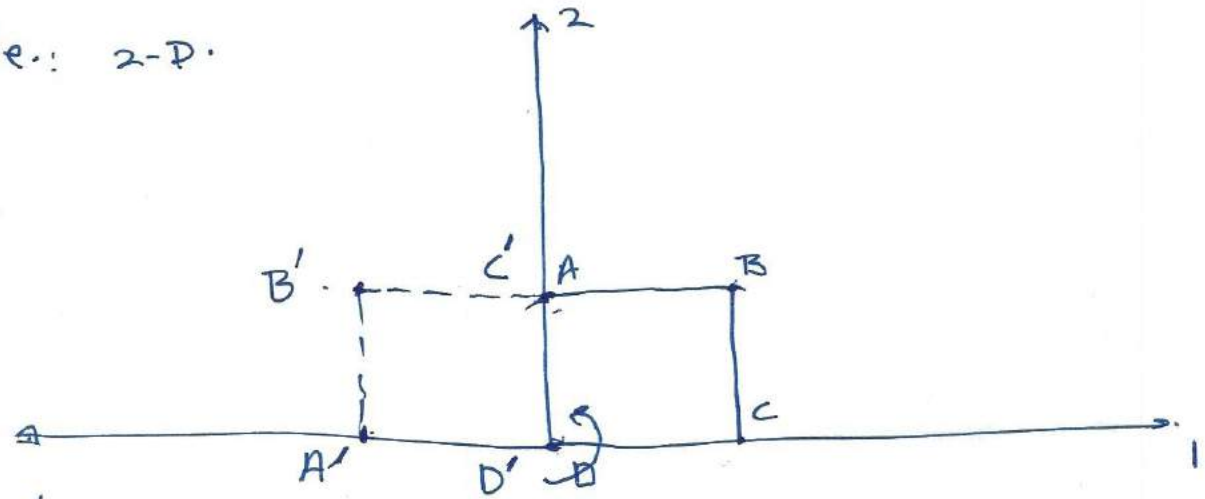
But $E_{ij} = 0 \rightarrow \bar{\epsilon}_{ij}, \bar{\omega}_{ij} \neq 0$

In Eulerian system:

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right), \quad \omega_{ij} = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} - \frac{\partial u_i}{\partial x_j}\right)$$

$$e_{ij} = \epsilon_{ij} - \frac{1}{2}\left(\epsilon_{kj} - \omega_{kj}\right)\left(\epsilon_{ki} - \omega_{ki}\right)$$

Example:: 2-D.




Rotate about $x_3$ axis by $\frac{\pi}{2}$

$$x_1 = -x_2^0 \quad , \quad x_2 = x_1^0 \quad , \quad x_3 = x_3^0$$

$$u_1 = x_1 - x_1^0 \quad ; \quad u_2 = x_2 - x_2^0 \quad , \quad u_3 = 0$$

$$u_1 = -x_2^0 - x_1^0 \qquad u_2 = x_1^0 - x_2^0$$

$$\bar{\epsilon}_{ij} = \frac{1}{2}\left[\frac{\partial u_j}{\partial x_i^0} + \frac{\partial u_i}{\partial x_j^0}\right] \quad , \quad \bar{\omega}_{ij} = \frac{1}{2}\left[\frac{\partial u_j}{\partial x_i^0} - \frac{\partial u_i}{\partial x_j^0}\right]$$

$$\frac{\partial u_1}{\partial x_1^0} = -1 \quad , \quad \frac{\partial u_1}{\partial x_2^0} = -1 \quad , \quad \frac{\partial u_2}{\partial x_1^0} = 1 \quad , \quad \frac{\partial u_2}{\partial x_2^0} = -1$$

$$[\bar{\epsilon}_{ij}] = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad ; \quad [\bar{\omega}_{ij}] = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Can show :- $E_{ij} = 0$ ; $\tan \phi_3 = \infty \left(\phi_3 = \frac{\pi}{2}\right)$

For large displacement ; $\bar{\epsilon}_{ij} \neq 0$ can not be interpreted as presence of strain.

Exam 1: October 13 (Thurs), 14th Friday. (5 PM).

Thus far:-

(1) Deformation: $x_i = x_i(x_i^0)$ ; $x_i^0 = x_i^0(x_i)$

(2) Deformation gradient: $\frac{\partial x_i}{\partial x_j^0}$ ; $\frac{\partial x_i^0}{\partial x_j}$

Relates the behavior of neighboring points.

- Jacobian :- $J^0 = \det\left(\frac{\partial x_i}{\partial x_j^0}\right)$ , $J = \det\left(\frac{\partial x_i^0}{\partial x_j}\right)$

$0 < J^0 < \infty$ ; $0 < J < \infty$

(3) Deformation of

3.1 Lines : $d\hat{x} = \frac{\partial x_i}{\partial x_j^0} dx_j^0 \hat{e}_i$

3.2 Area : $dA_i = J^0 \frac{\partial x_K^0}{\partial x_i} dA_K^0$

3.3 Volume : $dV = J^0 dV^0$

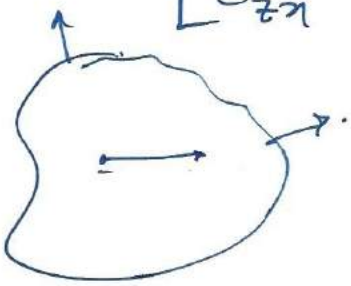
Density : $\rho^0 = J^0 \rho$

(4) Strain :

- Normal strain : $E_1 = \frac{ds - ds_0}{ds_0}$ , $e_1 = \frac{ds - ds_0}{ds}$
- Extension : $\frac{ds}{ds_0}$

strain :- $\begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ & \epsilon_{yy} & \\ \epsilon_{zx} & \epsilon_{zy} & \epsilon_{zz} \end{bmatrix}$ $\epsilon_{xx} = \frac{\Delta L}{L_0}$

$\epsilon_{xy} = \gamma_{xy}$.



- Tensorial strain measure:

$$E_{ij} = \frac{1}{2}\left[\frac{\partial x_m}{\partial x_i^0}\frac{\partial x_m}{\partial x_j^0} - \delta_{ij}\right], \quad e_{ij} = \frac{1}{2}\left[\delta_{ij} - \frac{\partial x_m^0}{\partial x_i}\frac{\partial x_m^0}{\partial x_j}\right]$$

Displacement : $u_i = x_i - x_i^0$

$$E_{ij} = \frac{1}{2}\left[\frac{\partial u_i}{\partial x_j^0} + \frac{\partial u_j}{\partial x_i^0} + \frac{\partial u_m}{\partial x_i^0}\frac{\partial u_m}{\partial x_j^0}\right]$$

$$e_{ij} = \frac{1}{2}\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{\partial u_m}{\partial x_i}\frac{\partial u_m}{\partial x_j}\right]$$

- Rotation :

$$\tan\phi_3 = \frac{\frac{1}{2}\left(\frac{\partial x_2}{\partial x_1^0} - \frac{\partial x_1}{\partial x_2^0}\right)}{\sqrt{\left(\frac{\partial x_1}{\partial x_1^0}\right)\left(\frac{\partial x_2}{\partial x_2^0}\right) - \frac{1}{4}\left(\frac{\partial x_1}{\partial x_2^0} + \frac{\partial x_2}{\partial x_1^0}\right)^2}}$$

$$\bar{\omega}_{ij} = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i^0} - \frac{\partial u_i}{\partial x_j^0}\right) \leftarrow \text{Rotation tensor.}$$

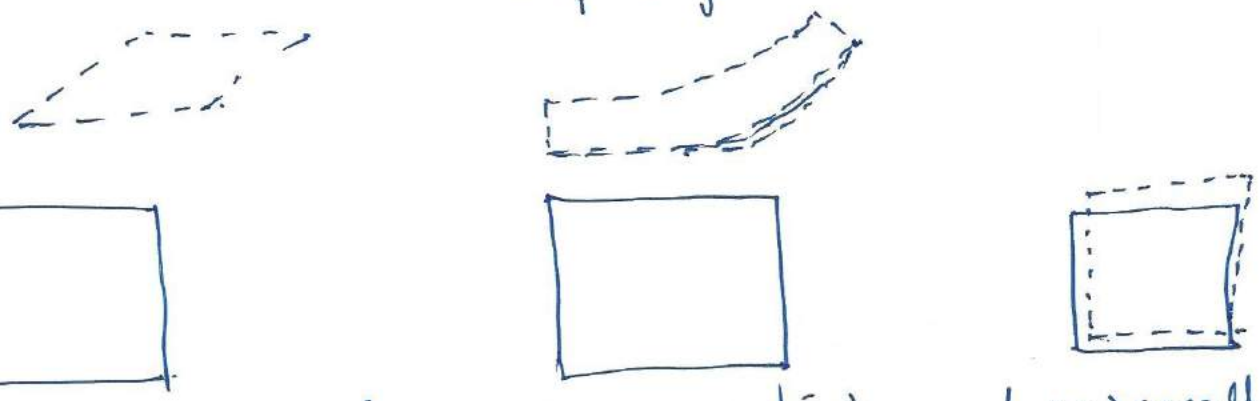
- $E_{ij} = \bar{\epsilon}_{ij} + \frac{1}{2}\left(\bar{\epsilon}_{kj} - \bar{\omega}_{kj}\right)\left(\bar{\epsilon}_{ki} - \bar{\omega}_{ki}\right)$

$$\bar{\epsilon}_{ij} = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i^0} + \frac{\partial u_i}{\partial x_j^0}\right); \quad \bar{\omega}_{ij} = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i^0} - \frac{\partial u_i}{\partial x_j^0}\right)$$

$$e_{ij} = \epsilon_{ij} - \frac{1}{2}(\epsilon_{kj} - \omega_{kj})(\epsilon_{ki} - \omega_{ki})$$

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right), \quad \omega_{ij} = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} - \frac{\partial u_i}{\partial x_j}\right).$$

Also: $$E_{ij} = e_{pq} \frac{\partial x_p}{\partial x_i^0} \frac{\partial x_q}{\partial x_j^0}$$



(a) Large displacements, Large rotations, Large deformations (strains).

(b) Large displacement, Large rotation, small strains → Shells, plates, beam: Non-linear analysis needed.

(c) small disp, rotation, strains

(1) If displacement gradient is small:

$$\left|\frac{\partial u_i}{\partial x_j^0}\right| \ll 1 \quad ; \quad \left|\frac{\partial u_i}{\partial x_j}\right| \ll 1$$

Ignore all product terms:

$$E_{ij} \approx \bar{\epsilon}_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j^0} + \frac{\partial u_j}{\partial x_i^0}\right)$$

$$e_{ij} \approx \epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

"Infinitesimal strain tensors"

(2) Deformations are small

$$E_1 \approx e_1 \; ; \; E_2 \approx e_2, \; E_3 \approx e_3$$

$$\alpha_{23} \approx \beta_{23}, \; \alpha_{13} \approx \beta_{13}, \; \alpha_{12} \approx \beta_{12}$$

This leads to $\bar{\epsilon}_{ij} \approx \epsilon_{ij}$

$$\bar{\epsilon}_{ij} \approx \epsilon_{ij} \simeq E_1 \simeq E_2 \simeq E_3.$$

$$\bar{\epsilon}_{12} \approx \bar{\epsilon}_{13} \approx \bar{\epsilon}_{23} \rightarrow \text{change of angle.}$$

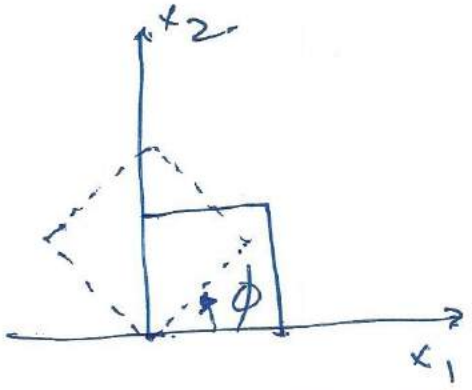
Additionally if $\left|\frac{u_i}{L}\right| \ll 1$ $\quad L \rightarrow$ smallest dimension

$x_i^0 \approx x_i \rightarrow$ Lagrangian & Eulerian frames become identical.

i.e. $\bar{\epsilon}_{ij} = \epsilon_{ij}$

2.6 Special deformations: (1-2 frame, unless specified different)

(I) Rigid body rotation by $\phi$




$$x_1 = x_1^0 \cos\phi - x_2^0 \sin\phi$$

$$x_2 = x_1^0 \sin\phi + x_2^0 \cos\phi$$

$$x_3 = x_3^0$$

$$\Rightarrow \quad u_1 = x_1^0(\cos\phi - 1) - x_2^0 \sin\phi$$

$$u_2 = x_1^0 \sin\phi + x_2^0(\cos\phi - 1)$$

$$u_3 = 0$$

$u_i = x_i - x_i^0$

$$\left[\frac{\partial u_i}{\partial x_j^0}\right] = \begin{bmatrix} \cos\phi - 1 & -\sin\phi & 0 \\ \sin\phi & \cos\phi - 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$E_{ij} = 0$$

$$\overline{\epsilon_{ij}} = \begin{bmatrix} \cos\phi - 1 & 0 & 0 \\ 0 & \cos\phi - 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \approx \frac{1}{2} \begin{bmatrix} -\frac{1}{2}\phi^2 & 0 & 0 \\ 0 & -\frac{1}{2}\phi^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\overline{\epsilon_{ij}} = 0$ iff $\phi$ is small.

(II) Homogeneous deformation:

Final position is a linear function of initial position.

$$\rightarrow \quad x_i = C_{iK} x_K^0 + B_i$$

$$\frac{\partial x_i}{\partial x_K^0} = C_{iK} = \text{constant}$$ i.e. deformation gradient is constant.

- The strain is the same irrespective of the location of point $x_i^0$.
- straight lines deform into straight lines. (but angles may change).
- All deformations can be treated as locally homogeneous.

(III) Uniform extension: (3-D).

$$x_1 = \lambda_1 x_1^\circ + b_1$$
$$x_2 = \lambda_2 x_2^\circ + b_2$$
$$x_3 = \lambda_3 x_3^\circ + b_3$$

$\lambda_1, \lambda_2, \lambda_3 \rightarrow$ stretches.

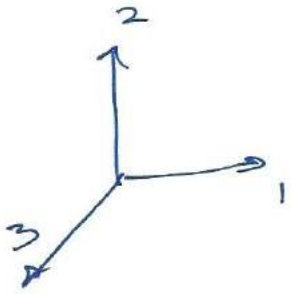
$b_1, b_2, b_3 \rightarrow$ rigid body motion.

uniaxial :- $\lambda_2 = \lambda_3 = \lambda_0$

$$u_1 = (\lambda_1 - 1) x_1^\circ + b_1$$
$$u_2 = (\lambda_2 - 1) x_2^\circ + b_2$$
$$u_3 = (\lambda_3 - 1) x_3^\circ + b_3$$

$$\left[\frac{\partial u_i}{\partial x_j^\circ}\right] = \begin{bmatrix} \lambda_1 - 1 & 0 & 0 \\ 0 & \lambda_2 - 1 & 0 \\ 0 & 0 & \lambda_3 - 1 \end{bmatrix}$$

$$[E_{ij}] = \begin{bmatrix} \frac{1}{2}(\lambda_1^2 - 1) & 0 & 0 \\ 0 & \frac{1}{2}(\lambda_2^2 - 1) & 0 \\ 0 & 0 & \frac{1}{2}(\lambda_3^2 - 1) \end{bmatrix}$$




Diagonal $\rightarrow$ principal strains

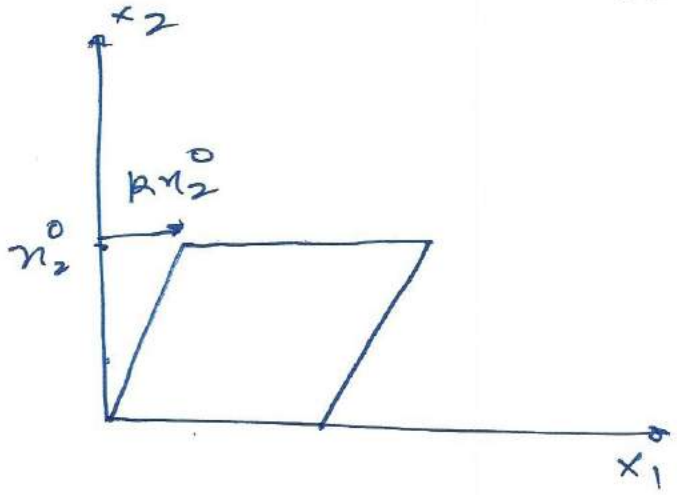
$x_1, x_2, x_3 \rightarrow$ principal directions

$$E_1 = \sqrt{1 + 2E_{11}} - 1 = \lambda_1 - 1$$

$$E_2 = \lambda_2 - 1 \quad , \quad E_3 = \lambda_3 - 1$$

Also : $\bar{\phi}_1 = \bar{\phi}_2 = \bar{\phi}_3 = 0$

[IV] Simple shear:




$$x_1 = x_1^0 + kx_2^0$$
$$x_2 = x_2^0$$
$$x_3 = x_3^0$$

$\therefore u_1 = kx_2^0, \quad u_2 = 0, \quad u_3 = 0$

$$E_{ij} = \begin{bmatrix} 0 & \frac{k}{2} & 0 \\ \frac{k}{2} & \frac{k^2}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

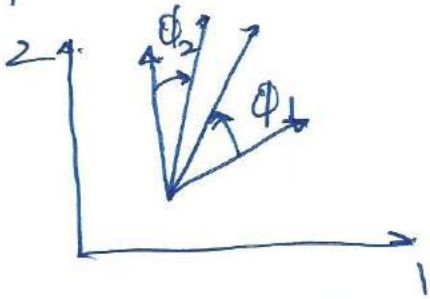
$E_{22} \neq 0.$

$E_2 = \sqrt{1+k^2} - 1$

$\phi_1 = 0, \quad \phi_2 = 0, \quad \tan\phi_3 = \dfrac{-\frac{1}{2}k}{\sqrt{1-\frac{k^2}{4}}}$

at $k=2$, $\phi_3 = -\frac{\pi}{2}$. ← Flattening.

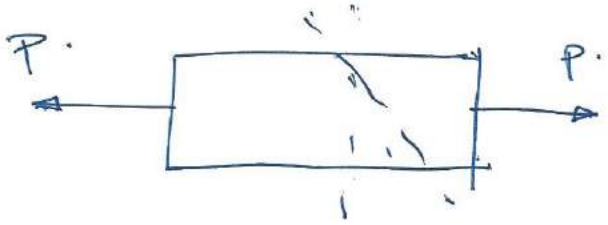
(V) Generalized Homogeneous Deformation:




$$x_1 = (1+E_1)\cos\phi_1 x_1^0 + (1+E_2)\sin\phi_2 x_2^0$$
$$x_2 = (1+E_1)\sin\phi_1 x_1^0 + (1+E_2)\cos\phi_2 x_2^0$$
$$x_3 = x_3^0$$

$$[E_{ij}] = \begin{bmatrix} E_1 + \frac{1}{2}E_1^2 & \frac{1}{2}(1+E_1)(1+E_2)\sin(\phi_1+\phi_2) & 0 \\ \frac{1}{2}(1+E_1)(1+E_2)\sin(\phi_1+\phi_2) & E_2 + \frac{1}{2}E_2^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

# Chapter 3 : Stress.




Normal stress $= \frac{P}{A}$ $\frac{N}{m^2}$

↓

"Traction"

"Stress vector"

Three types of forces :-

(1) Extrinsic :- arise from outside the body.

Eg: Gravity, Electro-magnetic forces

(2) Intrinsic forces :- arise from within and act upon pair of particles.

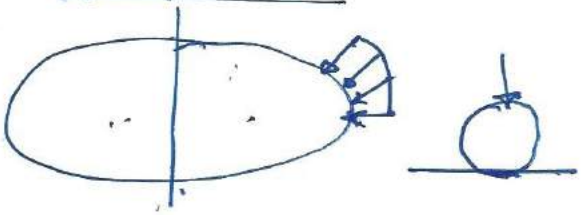
Eg: Newtonian gravity.

(3) Contact force :- act upon bounding surfaces. ~~are~~ and are equipollent.

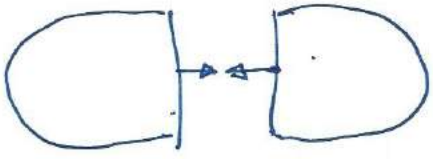
Equipollent forces

↓

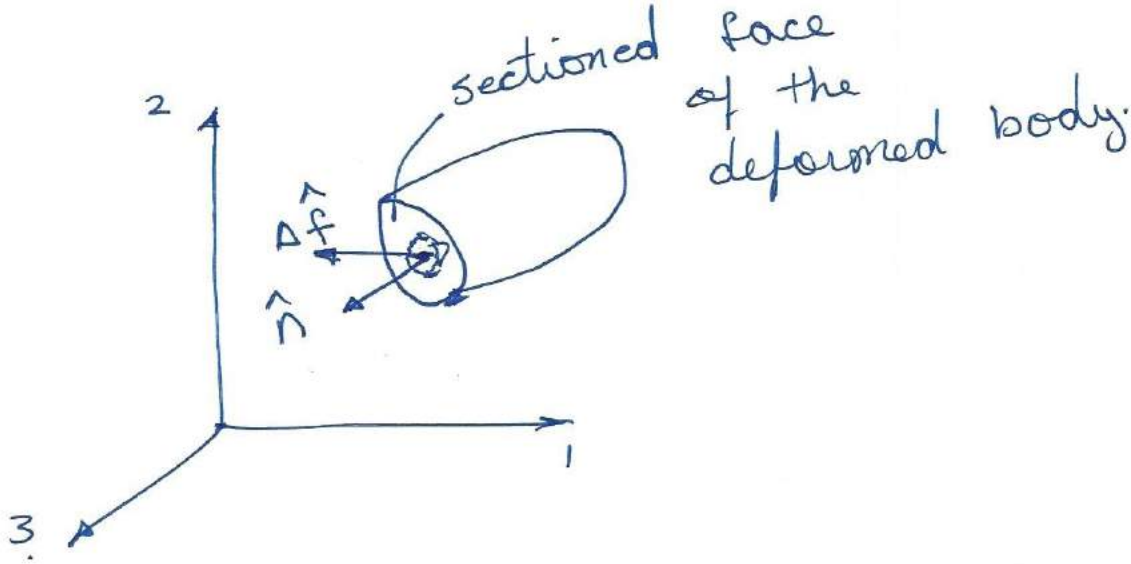
Pressure, stress.



- Surface forces, tractions

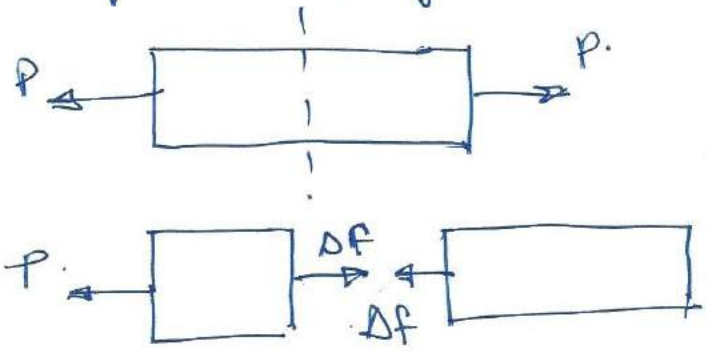


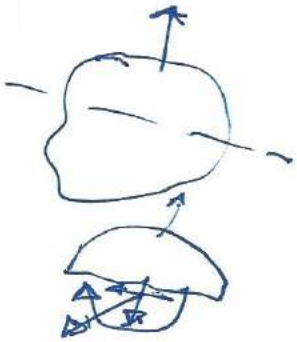
## 3.1 Cauchy stress Principle




$\hat{n}$ → unit vector perpendicular to the area.

$\Delta\hat{f}$ → resultant force due to the remaining part of the body.

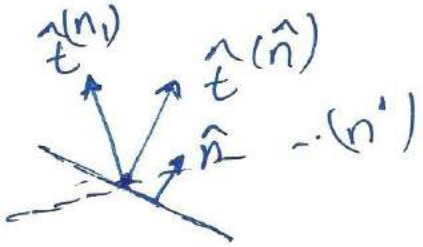


For a continuous body:

$$\lim_{\Delta A \to 0} \frac{\Delta \hat{F}}{\Delta A} = \frac{d\hat{f}}{dA} = \hat{t}^{(\hat{n})}$$

$\hat{t}^{(\hat{n})}$ Traction vector.



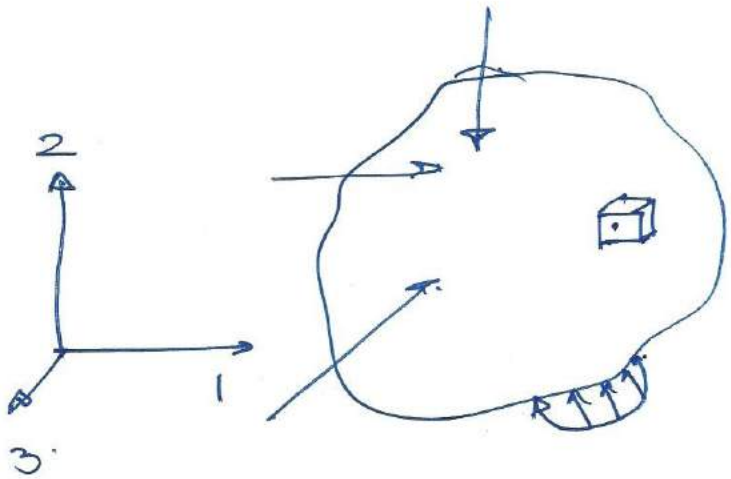

depends on the orientation of the area.

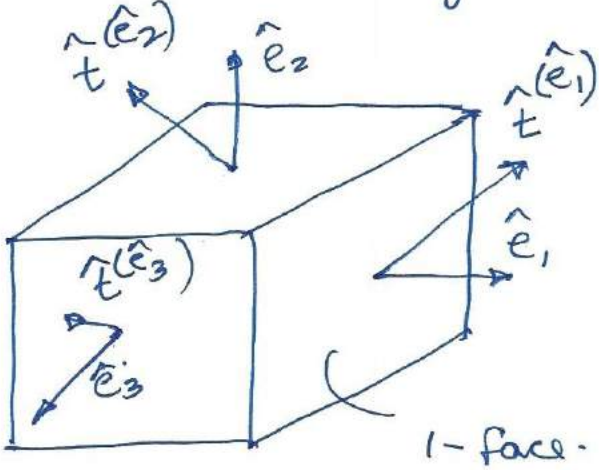
In general, there are moments & torsions acting on $\Delta A$

$$\hat{m}^{(\hat{n})} = \lim_{\Delta A \to 0} \frac{\Delta \hat{m}}{\Delta A} = 0$$

i.e. we ignore couple stresses.

— Microstructural → Couple stress theory.




Traction on 1- face :- $\hat{n} = \hat{e}_1$

$$\hat{t}^{(\hat{n})} = \hat{t}_i^{(\hat{e}_1)} \hat{e}_i = t_1^{(\hat{e}_1)} \hat{e}_1 + t_2^{(\hat{e}_1)} \hat{e}_2 + t_3^{(\hat{e}_1)} \hat{e}_3$$

on 2- face :- $\hat{n} = \hat{e}_2$

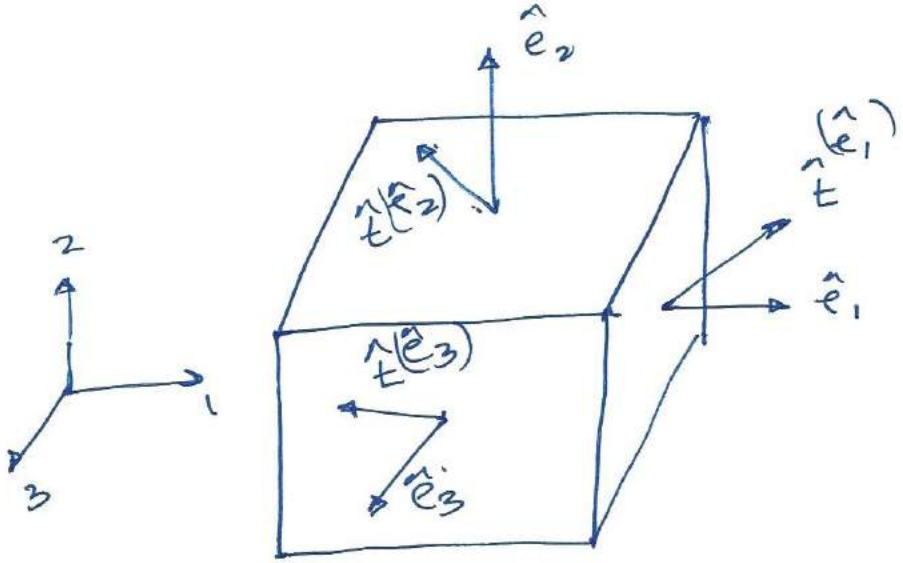
$$\hat{t}^{(\hat{n})} = t_i^{(\hat{e}_2)} \hat{e}_i = t_1^{(\hat{e}_2)} \hat{e}_1 + t_2^{(\hat{e}_2)} \hat{e}_2 + t_3^{(\hat{e}_2)} \hat{e}_3$$

in general :- $t_1^{(\hat{e}_2)} \neq t_2^{(\hat{e}_1)}$

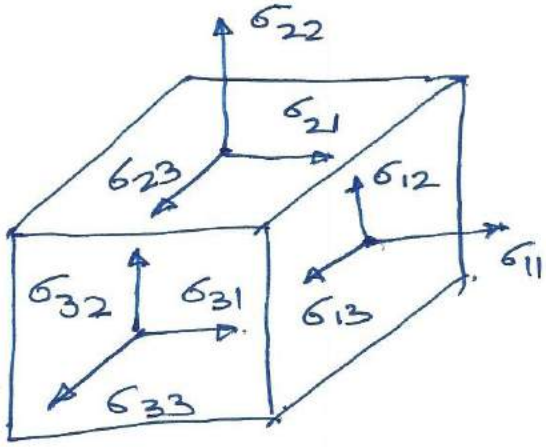
Introduce a new notation :-

$$\sigma_{ij} = t_j^{(\hat{e}_i)} \qquad i \to \text{face}, \quad j \to \text{component}$$

$$\sigma_{11} = t_1^{\hat{e}_1}, \quad \sigma_{13} = t_3^{\hat{e}_1}$$



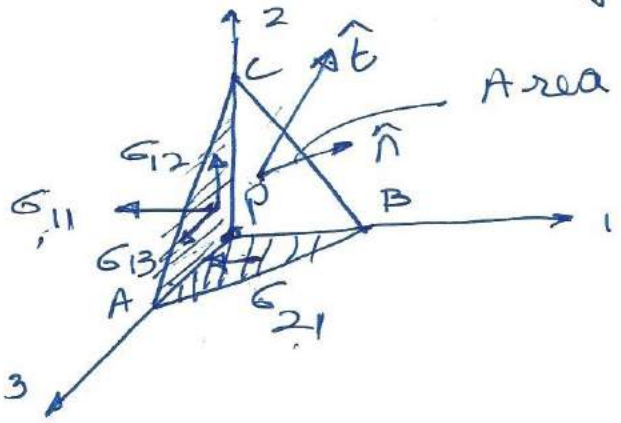

Tractions.




Projections.

$\hat{t} = \frac{\text{Force}}{\text{area}}$ ; vector → 3 components & it transforms as a vector.

→. Relation between $t_i$ & $n_j$




Area of the sectioned face is dA ↓ (ABC)

Equilibrium

$$t_1 dA - \sigma_{11} dA_1 - \sigma_{21} dA_2 - \sigma_{31} dA_3 + b_1 \rho dV = 0$$

$b_1$ → $x_1$ component of a body force

$t_1$ →. $x_1$ component of the traction vector

$dA_i$ → Area of the face perpendicular to i-direction

$dA \rightarrow$ area of the inclined surface.

$dV \rightarrow$ volume of the tetrahedron.

$dV = \frac{1}{3} h\, dA$ , $h$ is the smallest distance from P to the area $dA$.

$$\hat{n} = n_1 \hat{e}_1 + n_2 \hat{e}_2 + n_3 \hat{e}_3$$

$dA_1 = dA\, n_1$ $\rightarrow$ area of face 1

$dA_2 = n_2\, dA$ area of face 2

$dA_3 = n_3\, dA$ area of face 3.

$$t_1\, dA = \sigma_{11}\, n_1 dA + \sigma_{21}\, n_2 dA + \sigma_{31}\, n_3 dA - b_1 \rho \tfrac{1}{3} h\, dA$$

Let $h \rightarrow 0$, divide by $dA$.

$$t_1 = \sigma_{11} n_1 + \sigma_{21} n_2 + \sigma_{31} n_3 = \sigma_{j1} n_j$$

Equilibrium along 2 & 3 directions gives.

$$t_2 = \sigma_{12} n_1 + \sigma_{32} n_2 + \sigma_{33} n_3 = \sigma_{j2} n_j$$

$$t_3 = \sigma_{13} n_1 + \sigma_{23} n_2 + \sigma_{33} n_3 = \sigma_{j3} n_j$$

Cauchy formula :-

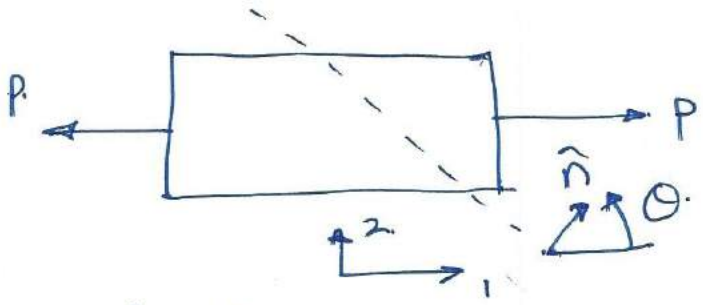
$$t_i = \sigma_{ji}\, n_j$$

$\downarrow$

Cauchy stress tensor.

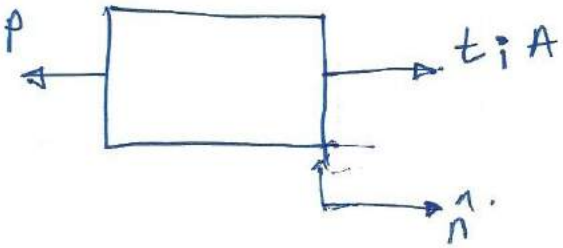
Example:-

For the given force distribution, find the traction on an interior plane inclined at an angle $\theta$.




$\rightarrow$ $t_i = \sigma_{ji} n_j$

Assume $\theta = 0$.




$\hat{n} = \{1, 0, 0\}$

$\hat{P} = \{P, 0, 0\}$

Apply equilibrium; $P_i = t_i A$

i.e. $t_i A = P_i$

i.e. $t_1 = \frac{P_1}{A}$, $t_2 = 0$, $t_3 = 0$

$t_i = \sigma_{ji} n_j$

$t_1 = \sigma_{11} n_1^{1} + \sigma_{21} n_2^{0} + \sigma_{31} n_3^{0} = \frac{P_1}{A} \begin{pmatrix} n_2 = 0 \\ n_3 = 0 \end{pmatrix}$

$t_2 = \sigma_{12} n_1 + \sigma_{22} n_2 + \sigma_{32} n_3$

$t_3 = \sigma_{13} n_1 + \sigma_{23} n_2 + \sigma_{33} n_3$

$t_1 = \sigma_{11} = \frac{P}{A}$; $t_2 = \sigma_{12} = 0$, $t_3 = \sigma_{13} = 0$

$\therefore$ $\sigma_{21} = 0$, $\sigma_{22} = 0$, $\sigma_{32} = 0$

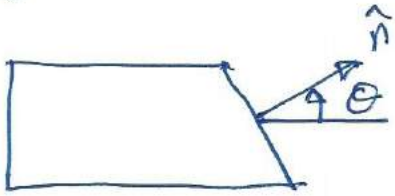
$\sigma_{31} = 0$, $\sigma_{23} = 0$, $\sigma_{33} = 0$

$$[\sigma_{ij}] = \begin{bmatrix} P/A & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

state of stress in the entire body.

For a cut made at angle $\theta$:

$$\hat{n} = \{\cos\theta, \sin\theta, 0\}$$



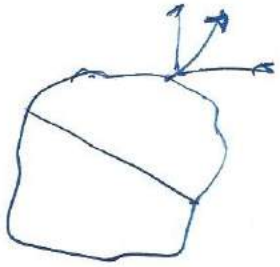

$$t_i = \sigma_{ji}\, n_j$$

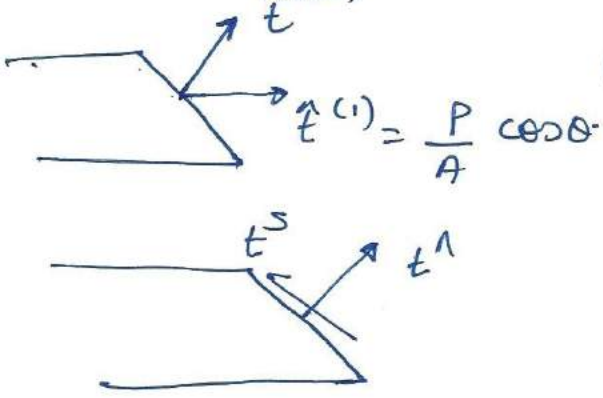
i.e. $t_1 = \sigma_{11} n_1 + \sigma_{21} n_2 + \sigma_{31} n_3$

$$= \frac{P}{A}\cos\theta + 0 \cdot \sin\theta + 0 \cdot 0$$

$$t_1 = \frac{P}{A}\cos\theta \quad , \quad t_2 = 0 \quad , \quad t_3 = 0$$

i.e. $\hat{t}^{(\theta)} = \left\{\frac{P}{A}\cos\theta, 0, 0\right\}$





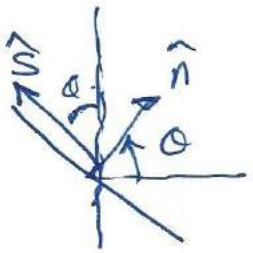




→ what are the normal and tangential components of the traction on a face at an angle $\theta$?

Normal :- $\hat{n} = \{\cos\theta, \sin\theta, 0\}$

Tangential :- $\hat{s} = \{-\sin\theta, \cos\theta, 0\}$




$$t_n = \hat{t}\cdot\hat{n} = t_k n_k = t_1 n_1 + t_2 n_2 + t_3 n_3$$

$$= \frac{P}{A}\cos\theta\cdot\cos\theta + 0 + 0$$

$$\boxed{t_n = \frac{P}{A}\cos^2\theta}$$

$$t_s = \hat{t}\cdot\hat{s} = t_k s_k = t_1 s_1 + t_2 s_2 + t_3 s_3$$

$$= -\frac{P}{A}\cos\theta\sin\theta + 0 + 0$$

$$\boxed{t_s = \frac{-P}{A}\cos\theta\sin\theta}$$

* Rotate $\sigma_{ij}$ by $\theta \rightarrow \sigma'_{ij}$

get $\sigma'_{11} = t_n$ , $\sigma'_{12} = t_s$

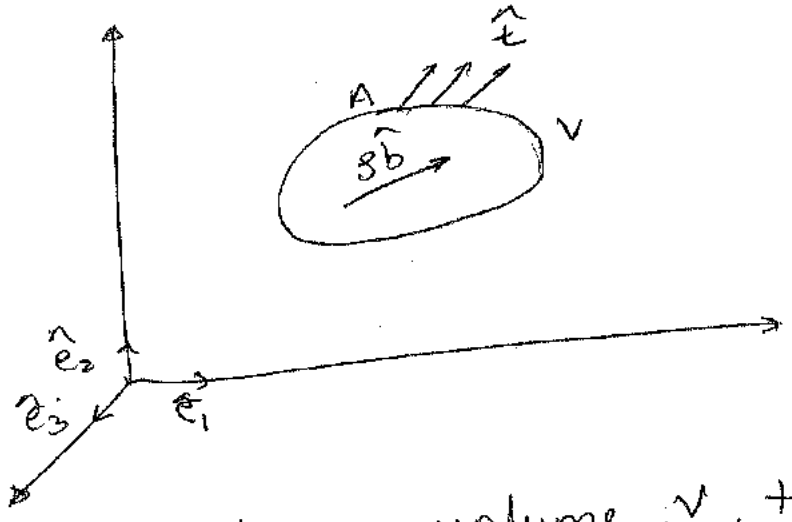
- Equations of motion in terms of stress..

For a rigid body, we can write the equations of motion as

$$\Sigma \hat{F} = m\hat{a}$$

$$\Sigma \hat{M} = m\hat{x} \times \hat{a}$$

$m \rightarrow$ mass
$\hat{a} \rightarrow$ accl.




Consider an arbitrary volume, V, taken from the deformed body. Then the Newton's laws of motion become,

$$\int_A \hat{t}\, dA + \int_V \rho\hat{b}\, dV = \int_V \rho\ddot{\hat{u}}\, dV$$

and

$$\int_A \hat{x} \times \hat{t}\, dA + \int_V \hat{x} \times \hat{b}\rho\, dV = \int_V \hat{x} \times \ddot{\hat{u}}\rho\, dV$$

where $\hat{t}$ is the traction on the boundary surface A, and $\hat{b}$ is the body force per unit mass.

In indicical notation, these are

$$\int_A t_i \, dA + \int_V \rho b_i \, dV = \int_V \rho \ddot{u}_i \, dV$$

$$\int_A \epsilon_{ijk} x_j t_k \, dA + \int_V \epsilon_{ijk} x_j b_k \rho \, dV = \int_V \epsilon_{ijk} x_j \ddot{u}_k \rho \, dV$$

These are equations of motion in terms of $t_i$.
By using $t_i = \sigma_{pi} n_p$ & by the integral theorem from chapter 1, we have

$$\int_A t_i \, dA = \int_A \sigma_{pi} n_p \, dA = \int_V \frac{\partial \sigma_{pi}}{\partial x_p} \, dV$$

Then, the equations of motion become

$$\int_V \left[ \frac{\partial \sigma_{pi}}{\partial x_p} + \rho b_i - \rho \ddot{u}_i \right] dV = 0$$

& $$\int_V \left[ \frac{\partial}{\partial x_p} (\epsilon_{ijk} x_j \sigma_{pk}) + \rho \epsilon_{ijk} b_k - \rho \epsilon_{ijk} x_j \ddot{u}_k \right] dV = 0$$

Noting that : $\frac{\partial}{\partial x_p}(x_j \sigma_{pk}) = \sigma_{jk} + x_j \frac{\partial \sigma_{pk}}{\partial x_p}$

we can expand the second equation and simplify it using the first equation. Further, since the volume V is arbitrary, we conclude that the integrands must vanish. Therefore

$$\frac{\partial \sigma_{pi}}{\partial x_p} + \rho b_i = \rho \ddot{u}_i$$

$$\epsilon_{ijk} \sigma_{jk} = 0$$

The second equation shows that $\sigma_{ij}$ is a symmetric tensor. [since contraction of a symmetric tensor ($\sigma_{ij}$) with an anti-symmetric tensor ($\epsilon_{ijk}$) is zero].

Thus the two equations of motion become

$$\frac{\partial \sigma_{ij}}{\partial x_j} + \rho b_i = \rho \ddot{u}_i$$

&

$$\sigma_{ij} = \sigma_{ji}$$

In expanded notation:

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{21}}{\partial x_2} + \frac{\partial \sigma_{31}}{\partial x_3} + \rho b_1 = \rho \ddot{u}_1$$

$$\frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{32}}{\partial x_3} + \rho b_2 = \rho \ddot{u}_2$$

$$\frac{\partial \sigma_{13}}{\partial x_1} + \frac{\partial \sigma_{23}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} + \rho b_3 = \rho \ddot{u}_3$$

Its worth repeating that due to the symmetry of stress, only 6 components are independent.

$\therefore \sigma_{12} = \sigma_{21}$, $\sigma_{13} = \sigma_{31}$, and $\sigma_{23} = \sigma_{32}$.

Also, to keep repeating the point, these equations in terms of Cauchy stress measure are applicable to large deformations.

Example: Under what circumstances (if any) is the given symmetric stress field in ~~symmetric~~ static equilibrium?

$$\sigma_{11} = 3x_1 + k_1 x_2^2 \quad , \quad \sigma_{22} = 2x_1 + 4x_2$$

$$\sigma_{12} = \sigma_{21} = a + bx_1 + cx_1^2 + dx_2 + ex_2^2 + fx_1x_2 \,,$$

all others being zero.

→ Since no information about body forces is given, we assume they are zero. same thing with inertia. Also, since $\sigma_{13} = \sigma_{32} = \sigma_{33} = 0$, ignore the 3-direction & treat it as a 2-D problem. So, only first two equilibrium equation remain.

Solving, we get

$$3 + d + 2ex_2 + fx_1 = 0$$

and $$4 + b + 2cx_1 + fx_2 = 0$$

This should be true for any $x_1, x_2$ which leads to

$3 + d = 0$ i.e. $d = -3$

$e = 0$

$f = 0$

$c = 0$

$4 + b = 0$ i.e. $b = -4$.

Isn't it interesting that $k_1$ term doesn't effect the equilibrium!!

## 3-2 Properties of the Cauchy stress tensor.

Since $\sigma_{ij}$ is symmetric and second order, it inherits many properties as already outlined in chapter 1 for 2nd order tensors. It is worthwhile to review these in the physical context of stress.

- The Cauchy Stress Tensor:

Let's recapitulate the development of the Cauchy stress tensor.

On any surface of a deformed body, there exists a traction vector $\hat{t}$. The magnitude and direction of this vector at a point changes on different surfaces with different unit vectors $\hat{n}$.

The 3-components of the stress vector on $x_1$-face (the one that is perpendicular to $x_1$-axis) are denoted as

$$\sigma_{11}, \sigma_{12}, \sigma_{13}$$

Similarly on $x_2$- & $x_3$- faces, we have

$$\sigma_{21}, \sigma_{22}, \sigma_{23} \quad \text{and} \quad \sigma_{31}, \sigma_{32}, \sigma_{33}, \text{ respectively}$$

Note that although these nine scalars $[\sigma_{ij}]$ have the size of a 2nd order tensor, they cannot be assumed automatically to have formed a tensor, as they have not yet shown to satisfy any co-ordinate transformation law.

Recall from chapter 1, the quotient rule says if we have $C_{ijK} D_K = A_{ij}$, and we know that $C_{ijK}$ is a tensor and $A_{ij}$ is also a tensor, then it can be shown that $D_K$ is a tensor. For $\sigma_{ij}$ we have: $t_j = \sigma_{ji} n_j$

Here, $t_i$ is a tensor of order 1, and $n_j$ is a tensor of order 1, thus by quotient rule, $\sigma_{ij}$ is a 2nd order tensor. i.e. $\sigma'_{ij} = \beta_{iM} \beta_{jN} \sigma_{MN}$

The state of stress in a body is completely given by the stress tensor $\sigma_{ij}$. Thus, as we showed in example 1, given any surface with the unit normal vector $\hat{n}$, one is able to determine the traction (or "stress vector", or "force intensity") acting on that surface if the stress tensor is known. Finally, we obtained the equations of motion of an arbitrary volume from which we concluded that $\sigma_{ij}$ is symmetric.

In this development, the physical entity is the traction vector; the stress tensor is viewed as a derived (or defined) quantity that connects the components of the traction vector acting on any plane.

## Principal Stresses

In general, the traction vector $\hat{t}$ acting on a surface depends on $\hat{n}$, but is not parallel to $\hat{n}$.

$\hat{t} \overset{d}{\parallel} \hat{n}$

However, if $\hat{t}$ is acting in the direction of $\hat{n}$, then we can write

$$t_i = \sigma n_i \quad , \text{ where } \sigma \text{ is the magnitude (scalar) of the traction vecto}$$

By using the relation $t_i = \sigma_{ji} n_j$, the above yields

$$t_i = \sigma_{ji} n_j = \sigma n_i$$

which can be rewritten as

$$\sigma_{ji} n_j = \sigma n_i$$

i.e. $$\left[\sigma_{ji} - \sigma \delta_{ij}\right]^T \{n_j\} = 0$$

These are three equations for three unknowns $n_1, n_2, n_3$. For this system of equations to yield a non-trivial solution, we need

$$\det\left[\sigma_{ji} - \sigma \delta_{ij}\right] = 0$$

This is a standard eigenvalue problem. The direction which satisfies this is called a principal direction of stress, and $\sigma$ is the correspondin

principal stress. If $\sigma_{ij}$ ~~is~~ a real, symmetric matrix, the eigenvalues $\sigma_i$ are guaranteed to be real, and the corresponding eigenvectors $n_i$ are mutually orthogonal. The proof for this theorem is available in any book on matrix theorem or linear algebra. [I recommend the book on linear algebra by Gilbert Strang.]

Example :- Consider we are given $[\sigma_{ij}] = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

We can calculate the principal stresses using eigenvector method. Or, we can calculate the stress invariants as $I_1 = 5$, $I_2 = 4$, and $I_3 = 0$. The characteristic equation is,

$$\sigma^3 - 5\sigma^2 + 4\sigma = 0$$

The principal values can be readily calculated as : $\sigma_1 = 4$, $\sigma_2 = 1$, and $\sigma_3 = 0$

The principal direction (unit normal) $\hat{n}^{(1)}$ corresponding to $\sigma_1 = 4$ can be obtained by substituting this value back into the system of equations to obtain

$$-2n_1^{(1)} + 2n_2^{(1)} = 0$$
$$2n_1^{(1)} - 2n_2^{(1)} = 0$$
$$-3n_3^{(1)} = 0$$

This indicates that the three equations are not independent. Thus, only two equations are available to determine the solution.

However, since $\hat{n}$ is a unit vector, we have

$$n_1^2 + n_2^2 + n_3^2 = 1$$

This works as the third independent equation. The solution is

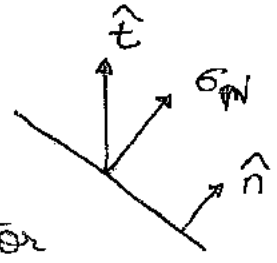
$$n_1^{(1)} = \frac{1}{\sqrt{2}}, \quad n_2^{(1)} = \frac{1}{\sqrt{2}}, \quad n_3^{(1)} = 0$$

Similarly, we can get the unit vectors $\hat{n}^{(2)}$ and $\hat{n}^{(3)}$ corresponding to $\sigma_2$ and $\sigma_3$. They are

$$\hat{n}^{(2)} = \{0, 0, 1\}, \text{ and } \hat{n}^{(3)} = \left\{\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, 0\right\}$$

It is straightforward to show that $\hat{n}^{(i)}$ are orthogonal.

- Normal stress:




The normal stress, $\sigma_N$, is the component of the traction vector in the direction of $\hat{n}$, the unit normal to the surface of interest. We can obtain this component of

stress by taking the scalar product of $\hat{t}$ and $\hat{n}$ as

$$\sigma_N = t_i n_i = \sigma_{ij} n_i n_j$$

If the coordinate axes are chosen to coincide with the principal directions of stress, then $\sigma_{ij}$ has the form of a diagonal matrix,

$$\text{i.e.} \quad \sigma_{ij} = 0 \quad \text{if } i \neq j$$

and

$$\sigma_{11} = \sigma_1 \;, \quad \sigma_{22} = \sigma_2 \;, \quad \sigma_{33} = \sigma_3$$

In that case, we can express $\sigma_N$ as

$$\sigma_N = \sigma_1 n_1^2 + \sigma_2 n_2^2 + \sigma_3 n_3^2$$

If the principal stresses are ordered such that $\sigma_1 \geq \sigma_2 \geq \sigma_3$, then we get

$$\sigma_1 > \sigma_N > \sigma_3$$

by using $n_1^2 + n_2^2 + n_3^2 = 1$. This indicates that $\sigma_1$ & $\sigma_3$ are the maximum and minimum normal stresses, respectively, at the point.

We can formally establish this by using $\sigma_N = \sigma_{ji} n_i n_j$ and when finding the extremum we need to take into account that $n_K n_K = 1$. We implement this by using a Lagrange multiplier and extremize

$$\phi = \sigma_{ji} n_j n_i - \lambda n_K n_K$$

where $\lambda$ (as yet unknown) is the Lagrange multiplier. Differentiating this with respect to $n_p$ leads to

$$\sigma_{ji} n_j - \lambda n_i = 0$$

i.e. $$[\sigma_{ji} - \lambda \delta_{ij}]^T \{n_j\} = 0$$

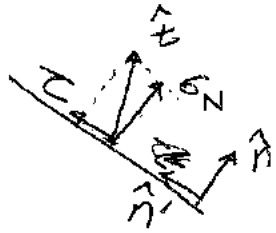
We now recognize $\lambda$ as the principal value of stress. Thus $\sigma_1$ and $\sigma_3$ are the maximum and minimum normal stresses.

- Shear Stress

Similar to normal stress, we can obtain shearing stress as the projection, $\tau$, of the stress vector on the surface of interest.

i.e. we can write

$$\hat{t} = \sigma_N \hat{n} + \tau \hat{n}'$$




or $$\tau \hat{n}' = \hat{t} - \sigma_N \hat{n}$$

where $\hat{n}'$ is perpendicular to $\hat{n}$. The magnitude of $\tau$ can be obtained as

$$\tau^2 = (\tau \hat{n}') \cdot (\tau \hat{n}')$$

$$= (\hat{t} - \sigma_N \hat{n}) \cdot (\hat{t} - \sigma_N \hat{n})$$

$$= |\hat{t}|^2 - \sigma_N^2$$

$|\hat{t}|$ → magnitude of $\hat{t}$.

$$\tau^2 = t_i t_i - \sigma_N^2$$

If we choose the principal directions as the co-ordinate axes, then

$$t_1 = \sigma_{ij} n_j = \sigma_1 n_1$$
$$t_2 = \sigma_{2j} n_j = \sigma_2 n_2$$
$$t_3 = \sigma_{3j} n_j = \sigma_3 n_3$$

Thus,

$$|\hat{t}|^2 = t_i t_i = t_1^2 + t_2^2 + t_3^2$$
$$= (\sigma_1 n_1)^2 + (\sigma_2 n_2)^2 + (\sigma_3 n_3)^2$$

$$\therefore \quad \tau^2 = (\sigma_1 n_1)^2 + (\sigma_2 n_2)^2 + (\sigma_3 n_3)^2 - (\sigma_1 n_1^2 + \sigma_2 n_2^2 + \sigma_3 n_3^2)$$

$$\downarrow$$

$$(\because \sigma_N = \sigma_1 n_1^2 + \sigma_2 n_2^2 + \sigma_3 n_3^2)$$

Expanding and using $n_i n_i = 1$ to replaces terms such as $1 - n_1^2 = n_2^2 + n_3^2$, we obtain

$$\tau^2 = n_1^2 n_2^2 (\sigma_1 - \sigma_2)^2 + n_2^2 n_3^2 (\sigma_2 - \sigma_3)^2 + n_3^2 n_1^2 (\sigma_3 - \sigma_1)^2$$

i.e. on surfaces with

$n_1 = 1$, if $n_2 = n_3 = 0 \implies \tau = 0$

$n_2 = 1$, if $n_1 = n_3 = 0 \implies \tau = 0$

$n_3 = 1$, $n_2 = n_1 = 0 \implies \tau = 0$.

while $\sigma_N$ reaches the max or min values (as shown in previous section)

Thus, $\tau$ is zero along principal directions!

Now consider all surfaces that contain the $x_2$-axis with

$$n_1 \neq 0 \ , \ n_3 \neq 0 \ , \ n_2 = 0$$

Then, the shear stress reduces to

$$\tau^2 = n_3^2 n_1^2 (\sigma_3 - \sigma_1)^2$$

$$= (1 - n_1^2) n_1^2 (\sigma_3 - \sigma_1)^2 \qquad (\because n_1^2 + n_3^2 = 1)$$

To obtain the extremum value of $\tau$,

$$\frac{\partial(\tau^2)}{\partial n_1} = 0 = (2n_1 - 4n_1^3)(\sigma_3 - \sigma_1)^2$$

Solving for $n_1$ gives us the directions

$$n_1 = \pm \frac{1}{\sqrt{2}} \ , \ n_2 = 0 \ , \ n_3 = \pm \frac{1}{\sqrt{2}}$$

i.e. the maximum shear stress occurs on the surfaces bisecting the angle between the $x_1$- and $x_3$- axes. The corresponding value of the shearing stress is

$$\tau_{max}^2 = \frac{1}{4}(\sigma_3 - \sigma_1)^2$$

$$\text{i.e. } \tau_{max} = \frac{1}{2}|\sigma_3 - \sigma_1|$$

We can use these values of principal & shear stresses to draw the Mohr's circle, similar to the procedure used in elementary mechanics of materials courses.

## 3-3 Stresses referred to the undeformed state.

So far, we have derived all stress equations by looking at the deformed configuration. To complete our duality of treatments of stress and deformation we need to consider the equations of motion with respect to the undeformed configuration. Specifically, the Cauchy equations of motion are in terms of spatial partial derivatives and these must be changed to derivatives with respect to the undeformed state. While doing so, we introduce 2 new measures of stress, the Lagrangian and Kirchoff stresses, respectively.

### Equations of motion in undeformed state

In the deformed state, we got the equations of motion in terms of Cauchy stress tensor as

$$\frac{\partial \sigma_{ji}}{\partial x_j} + \rho b_i = \rho \ddot{u}_i$$

$$\sigma_{ij} = \sigma_{ji}$$

To obtain these in undeformed configuration, we begin with the body force $\hat{b}$, which is the body force per unit mass in the deformed configuration. We define the body force per unit mass in the undeformed state as $\hat{b}^0$ such that the resultant force is the same.

i.e. $b_i^0 \rho^0 dV^0 = b_i \rho dV$

By conservation of mass (i.e. $\rho^0 dV^0 = \rho dV$) we get

$$b_i^0 = b_i$$

This body force relation is also valid for inertial forces.

Next, we change the spatial derivatives to material derivatives (i.e. Lagrangian to Eulerian) as follows

$$\frac{\partial \sigma_{ji}}{\partial x_j} = \frac{\partial \sigma_{ji}}{\partial x_p^0}\frac{\partial x_p^0}{\partial x_j} = \frac{\partial}{\partial x_p^0}\left(\sigma_{ji}\frac{\partial x_p^0}{\partial x_j}\right) - \sigma_{ji}\frac{\partial^2 x_p^0}{\partial x_p^0 \partial x_j} = \frac{\partial}{\partial x_p^0}\left(\sigma_{ji}\frac{\partial x_p^0}{\partial x_j}\right)$$

Thus, the equations of motion become

$$\frac{\partial}{\partial x_p^0}\left(\sigma_{ji}\frac{\partial x_p^0}{\partial x_j}\right) + \rho b_i = \rho \ddot{u}_i$$

Recall: $\rho/\rho^0 = J$ and $\frac{\partial J}{\partial x_p^0} = 0$, we get

$$\frac{\partial}{\partial x_p^0}\left(\frac{\rho^0}{\rho}\sigma_{ji}\frac{\partial x_p^0}{\partial x_j}\right) + \rho^0 b_i = \rho^0 \ddot{u}_i$$

Now, we need to replace the term in the parenthesis with a quantity that has meaning in the undeformed configuration.

I: Lagrangian stress:

The simplest approach is to define a new stress as

$$\sigma^L_{Pi} = \frac{\rho^0}{\rho}\,\sigma_{ji}\,\frac{\partial x^0_P}{\partial x_j}$$

Then, we can write

$$\frac{\partial\,\sigma^L_{Pi}}{\partial x^0_P} + \rho^0 b^0_i = \rho^0 \ddot{u}_i$$

and the symmetry condition $\sigma_{ij} = \sigma_{ji}$ is now given by

$$\sigma^L_{Pi}\,\frac{\partial x_j}{\partial x^0_P} = \sigma^L_{Pj}\,\frac{\partial x_i}{\partial x^0_P}$$

indicating that $\sigma^L_{Pi}$ is non-symmetric.

Because of the role played by $\sigma^L_{Pi}$, we can interpret it as a stress tensor. We call it the <u>Lagrange stress tensor</u>. Using this new stress, the equations of motion are relatively simple (actually, they resemble Cauchy equations), but the non-symmetric components will complicate the constitutive relation.

II: Kirchoff stress

If we prefer a symmetric stress tensor, then we can replace the term in the paranthesis with

$$\frac{\partial x_i}{\partial x_j^\circ}\,\sigma_{kj}^{K} \equiv \frac{\rho^\circ}{\rho}\,\sigma_{ji}\,\frac{\partial x_p^\circ}{\partial x_j}$$

from which we have

$$\sigma_{pq}^{K} = \frac{\rho^\circ}{\rho}\,\sigma_{ji}\,\frac{\partial x_p^\circ}{\partial x_i}\,\frac{\partial x_q^\circ}{\partial x_j}$$

It is evident that $\sigma_{ij}^{K} = \sigma_{ji}^{K}$ making $\sigma_{ij}^{K}$ a symmetric tensor. The equations of motion then become

$$\frac{\partial}{\partial x_K^\circ}\left[\frac{\partial x_i}{\partial x_j^\circ}\,\sigma_{kj}^{K}\right] + \rho^\circ b_i^\circ = \rho^\circ \ddot{u}_i$$

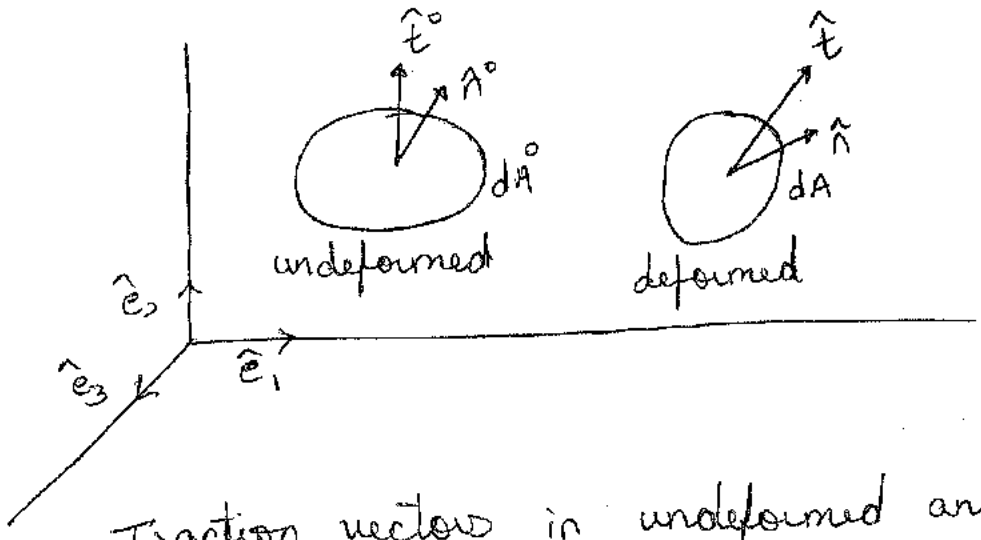
Again, because of the role played by $\sigma_{pi}^{K}$ we can interpret it as a stress tensor and it is called the Kirchoff stress tensor or second Piola stress tensor. The Kirchoff stress tensor is symmetric which simplifies the constitutive relation, but the equations of motion are slightly more complicated.

## Interpretation of Lagrangian and Kirchoff stresses

Since true loading exists only in the deformed state, the corresponding loading and stress in the body at the initial (undeformed) state could be considered as fictitious. However, we showed that even the Cauchy stress is a derived quantity - so, we are free to define other stresses! Let's first keep the following in mind:

- The traction vector is first defined in terms of a force divided by area.
- The stress tensor is defined according to a transformation relation for the traction and area normal.

To refer to the surface before deformation, we must define a traction vector $\hat{t}^\circ$ acting on an area $dA^\circ$. The introduction of such a vector is arbitrary and two usual definitions will be discussed.
But let's reconsider the Cauchy stress first to help understand the other two stresses.




Traction vectors in undeformed and deformed configurations.

I: Cauchy stress:

In the deformed state, on every plane surface passing through a point, there is a traction vector $t_i$ defined in terms of the deformed surface area. i.e. if the total resultant force acting on $dA$ is $d\hat{f}$, then we defined:

$$t_i \equiv \frac{df_i}{dA}$$

Mathematically, all traction vectors and unit normals in the deformed body form two vector spaces. Then the Cauchy stress tensor was shown to be the transformation between these two vector spaces,

i.e. $$t_i = \sigma_{ji} n_j$$

Defined in this manner, $\sigma_{ij}$ is an abstract quantity; however, on special plane surfaces (such as the ones with unit normals parallel to $\hat{e}_1$, $\hat{e}_2$, and $\hat{e}_3$) the

nine components of $\sigma_{ij}$ can be related to the traction vector and thus have physical meaning.

Thus, the meaning of $\sigma_{ij}$ are the components of stress derived from the force vector $df_i$ divided by the deformed area, $dA_i$. This, in elementary terms, is called "true stress".

II: Lagrangian stress

Let the resultant force $d\hat{f}^\circ$ referred to the undeformed configuration be identical to the force $d\hat{f}$ acting on the deformed area.

i.e. $$df_i^\circ = df_i$$

The Lagrangian traction vector, $t_i^\circ$, is defined as

$$t_i^\circ \equiv \frac{df_i^\circ}{dA^\circ} = \frac{df_i}{dA^\circ}$$

The Lagrangian stress tensor $\sigma_{ij}^L$ is defined as the second order tensor which relates the vector spaces $t_i^\circ$ and $n_i^\circ$ as

$$t_i^\circ = \sigma_{ji}^L n_j^\circ$$

Thus, the meaning of $\sigma_{pi}^L$ are the components of stress derived from the force $df_i$ divided by the original area. In elementary terms, this is called "engineering stress".

III. Kirchoff stress

Let the resultant force in the undeformed configuration not be the same as deformed configuration, but instead be given by a transformation as

$$df_i^\circ = \frac{\partial x_i^\circ}{\partial x_j} df_i$$

which follows the analogous rule as for the deformation of lines:

$$dx_j^\circ = \frac{\partial x_i^\circ}{\partial x_j} dx_j$$

* I re-emphasize: This is not a rotation transformation but that the force components (or line) are being 'deformed'

The Kirchoff stress ~~tensor~~ vector (i.e traction) is defined as

$$t_i^\circ \equiv \frac{df_i^\circ}{dA^\circ} = \frac{\partial x_i^\circ}{\partial x_j} \frac{df_j}{dA^\circ}$$

This leads to the definition of the Kirchoff stress tensor $\sigma_{ij}^K$:

$$t_i^\circ = \sigma_{ji}^K n_j^\circ$$

Formally, this is the same as for the Lagrangian stress, but recall that the components of force are different. The meaning of $\sigma_{pq}^K$ are components of ~~stress divided~~ stress derived from the transformed components of the force vector divided by original area. There is not elementary equivalent to this stress.

- Relations among the stress tensors

From the definition of the Lagrangian traction vector, we have

$$df_i = t_i^\circ dA^\circ = t_i dA$$

Using the Lagrangian and Cauchy stress tensors, the above equation can be written as

$$\sigma_{ji}^L n_j^\circ dA^\circ = \sigma_{pi} n_p dA$$

From chapter 2, we have

$$n_p dA = J^\circ \frac{\partial x_j^\circ}{\partial x_p} n_j^\circ dA^\circ$$

Then,

$$\sigma_{ji}^L n_j^\circ dA^\circ = J^\circ \sigma_{pi} \frac{\partial x_j^\circ}{\partial x_p} n_j^\circ dA^\circ$$

Thus:

$$\sigma_{ji}^L = J^\circ \frac{\partial x_j^\circ}{\partial x_p} \sigma_{pi} \qquad \text{where } J^\circ = \frac{\rho^\circ}{\rho}$$

i.e. $$\sigma_{ji}^L = \frac{\rho^\circ}{\rho} \frac{\partial x_j^\circ}{\partial x_p} \sigma_{pi}, \text{ or } \sigma_{ji} = \frac{\rho}{\rho^\circ} \frac{\partial x_j}{\partial x_p^\circ} \sigma_{pi}^L$$

Note that, in general, $\sigma_{ij}^L$ is not a symmetric tensor

Similarly we can relate the Kirchoff ~~stress~~ and Cauchy stress tensors as

$$\sigma_{ji}^K = \frac{\rho^\circ}{\rho} \frac{\partial x_i^\circ}{\partial x_m} \frac{\partial x_j^\circ}{\partial x_n} \sigma_{mn}, \quad \sigma_{ji} = \frac{\rho}{\rho^\circ} \frac{\partial x_i}{\partial x_m^\circ} \frac{\partial x_j}{\partial x_n^\circ} \sigma_{mn}^K$$

* In matrix form

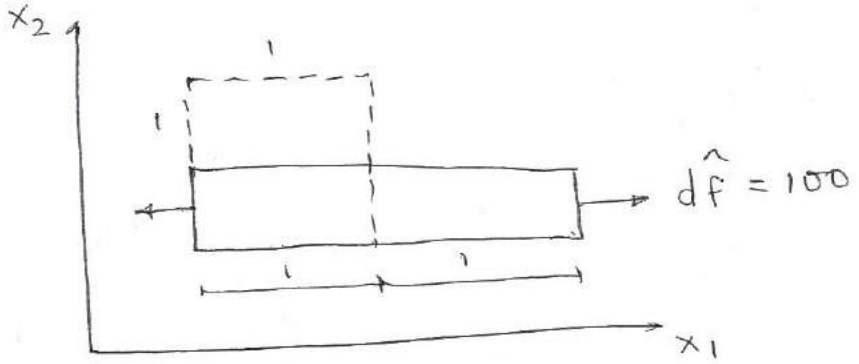
$$[\sigma_{ij}^K] = J \left[\frac{\partial x_i}{\partial x_m^\circ}\right]^{-1} [\sigma_{mn}] \left[\frac{\partial x_j^\circ}{\partial x_n}\right]^{-1^T}, \quad [\sigma_{ij}] = \frac{1}{J} \left[\frac{\partial x_i}{\partial x_m^\circ}\right] [\sigma_{mn}^K] \left[\frac{\partial x_j}{\partial x_n^\circ}\right]^T$$

Note that the Kirchoff stress tensor is symmetric. Since the strain tensors we introduced are symmetric, it is more convenient for us to use the Kirchoff stress tensor in the formulation of stress-strain laws.

# 3-4 Stress in special deformations

In chapter 2, we looked at some special deformations and the strains due to those deformations. Let us look at the stresses in three fundamental cases of simple extension, simple shear, and a state similar to bending. In the first, the magnitude of the areas change but not their orientation. In the ~~third~~ second and third, both magnitude and orientations of areas change.

1) Simple extension




In chapter 2, we saw that the deformation corresponding to a uniform extension can be written as:

$$x_1 = \lambda_1 x_1^\circ \quad , \quad x_2 = \lambda_2 x_2^\circ \quad , \quad x_3 = \lambda_3 x_3^\circ$$

Assume that the unit cubic solid shown above is subjected to simple extension where the applied load is acting in only one direction.

For this particular problem, the stretches are

$$\lambda_1 = 2, \quad \lambda_2 = \lambda_3 = 0.5 \ ;$$

and the basic information is given in terms of forces. So, we will establish the stresses using the connection between them, given forces, and the tractions.

We can obtain the deformation gradients as

$$\left[\frac{\partial x_i}{\partial x_p^\circ}\right] = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0.5 \end{bmatrix}, \quad \left[\frac{\partial x_p^\circ}{\partial x_i}\right] = \left[\frac{\partial x_i}{\partial x_p^\circ}\right]^{-1} = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

The Jacobian is : $J^\circ = 2 \times 0.5 \times 0.5 = 0.5$

i.e. there is a volume change.

A] Cauchy stress

The Cauchy stress tensor is obtained from information about the traction vectors;

$$t_i = \sigma_{ji} n_j = \frac{df_i}{dA} = \sigma_{1i} n_1 + \sigma_{2i} n_2 + \sigma_{3i} n_3$$

On $x_1$- face $\hat{n} = \hat{e}_1$ $(n_2 = 0, n_3 = 0)$

i.e. $t_i = \sigma_{1i}$

Also $df_i = \{100, 0, 0\}$, $dA_1 = \lambda_2 \lambda_3 = 0.5 \times 0.5 = 0.25$

$dA_2 = 0$

$dA_3 = 0$

similarly, on $x_2$ and $x_3$ faces, we have $t_i = \sigma_{2i}$ and $t_i = \sigma_{3i}$, respectively, and for both cases $df_i = \{0, 0, 0\}$

Thus, for the individual faces, we have

$$\hat{n}^{(1)} = \{1,0,0\}: \quad \sigma_{11} = \frac{df_1}{dA_1} = \frac{100}{0.25} = 400, \quad \sigma_{12} = \sigma_{13} = 0$$

$$\hat{n}^{(2)} = \{0,1,0\}: \quad \sigma_{21} = \sigma_{22} = \sigma_{23} = 0$$

$$\hat{n}^{(3)} = \{0,0,1\}: \quad \sigma_{31} = \sigma_{32} = \sigma_{33} = 0$$

i.e.

$$[\sigma_{ij}] = \begin{bmatrix} 400 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

We can use this and the deformation gradients to obtain the other stress definitions.

B] Lagrange stress:

$$\sigma^L_{Pi} = \frac{\rho^0}{\rho} \sigma_{ij} \frac{\partial x^0_P}{\partial x_j} = \frac{\rho^0}{\rho} \left[ \sigma_{i1} \frac{\partial x^0_P}{\partial x_1} + \sigma_{i2} \frac{\partial x^0_P}{\partial x_2} + \sigma_{i3} \frac{\partial x^0_P}{\partial x_3} \right]$$

Since $J^0 = \frac{\rho^0}{\rho} = \frac{1}{2}$, we can substitute all the values into the above expression. For example:

$$\sigma^L_{11} = \frac{1}{2}\left[400\left(\frac{1}{2}\right) + 0\times 0 + 0\times 0\right] = 100$$

$$\sigma^L_{12} = \frac{1}{2}\left[0\left(\frac{1}{2}\right) + 0\times 0 + 0\times 0\right] = 0$$

$$\vdots$$

so on.

We get:

$$[\sigma^L_{Pi}] = \begin{bmatrix} 100 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Note that its magnitude is different from Cauchy stress

c] Kirchoff stress:

$$\sigma_{pq}^{K} = \frac{\rho^0}{\rho}\sigma_{ij}\frac{\partial x_p^0}{\partial x_i}\frac{\partial x_q^0}{\partial x_j}$$

$$= J^0\left[\sigma_{11}\frac{\partial x_p^0}{\partial x_1}\frac{\partial x_q^0}{\partial x_1} + \sigma_{12}\frac{\partial x_p^0}{\partial x_1}\frac{\partial x_q^0}{\partial x_2} + \sigma_{13}\frac{\partial x_p^0}{\partial x_1}\frac{\partial x_q^0}{\partial x_3}\right.$$
$$+ \sigma_{21}\frac{\partial x_p^0}{\partial x_2}\frac{\partial x_q^0}{\partial x_1} + \sigma_{22}\frac{\partial x_p^0}{\partial x_2}\frac{\partial x_q^0}{\partial x_2} + \sigma_{23}\frac{\partial x_p^0}{\partial x_2}\frac{\partial x_q^0}{\partial x_3}$$
$$\left.+ \sigma_{31}\frac{\partial x_p^0}{\partial x_2}\frac{\partial x_q^0}{\partial x_1} + \sigma_{32}\frac{\partial x_p^0}{\partial x_3}\frac{\partial x_q^0}{\partial x_2} + \sigma_{33}\frac{\partial x_p^0}{\partial x_3}\frac{\partial x_q^0}{\partial x_3}\right]$$

Since only $\sigma_{11} \neq 0$,

$$\sigma_{pq}^{K} = J^0\sigma_{11}\frac{\partial x_p^0}{\partial x_1}\frac{\partial x_q^0}{\partial x_1}$$

And we get

$$\sigma_{11}^{K} = \frac{1}{2}(400)\left(\frac{1}{2}\right)^2 = 50$$

$$\sigma_{22}^{K} = \frac{1}{2}(400)(0) = 0$$

$$\sigma_{23}^{K} = \frac{1}{2}(400)(0) = 0$$

⋮

so on.

Thus:
$$[\sigma_{pq}^{K}] = \begin{bmatrix} 50 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

* We have original area $dA^0 = 1$ & deformed are $dA = 0.25$. Then Cauchy stress has the interpretation of force divided by deformed area, Lagrange stress has interpretation of force divided by original area, and Kirchoff stress is distorted force $df_i^0$ divided by original area $dA^0 = 1$

- Simple shear:

Consider the simple shear deformation given as

$$x_1 = x_1^0 + K x_2^0 \quad , \quad x_2 = x_2^0 \quad , \quad x_3 = x_3^0 .$$

Assume that the material constitutive behavior is given as:

$$\sigma_{ij} = 2\mu e_{ij} \quad , \text{ where } \mu \text{ is a modulus.}$$

A] Cauchy stresses:

Let's obtain the Cauchy stresses using the constitutive relationship, and the other stresses using the Cauchy stress.

The deformation gradient is

$$\left[\frac{\partial x_p}{\partial x_i^0}\right] = \begin{bmatrix} 1 & K & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} ; \left[\frac{\partial x_p^0}{\partial x_i}\right] = \left[\frac{\partial x_p}{\partial x_i^0}\right]^{-1} = \begin{bmatrix} 1 & -K & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$J = J^0 = 1$. Thus, there is no volume change.

We can calculate the Eulerian & Lagrangian strain tensors as

$$2e_{ij} = \delta_{ij} - \frac{\partial x_p^0}{\partial x_i} \frac{\partial x_p^0}{\partial x_j} = \begin{bmatrix} 0 & K & 0 \\ K & -K^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$2E_{ij} = \frac{\partial x_p}{\partial x_i^0} \frac{\partial x_p}{\partial x_j^0} - \delta_{ij} = \begin{bmatrix} 0 & K & 0 \\ K & K^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

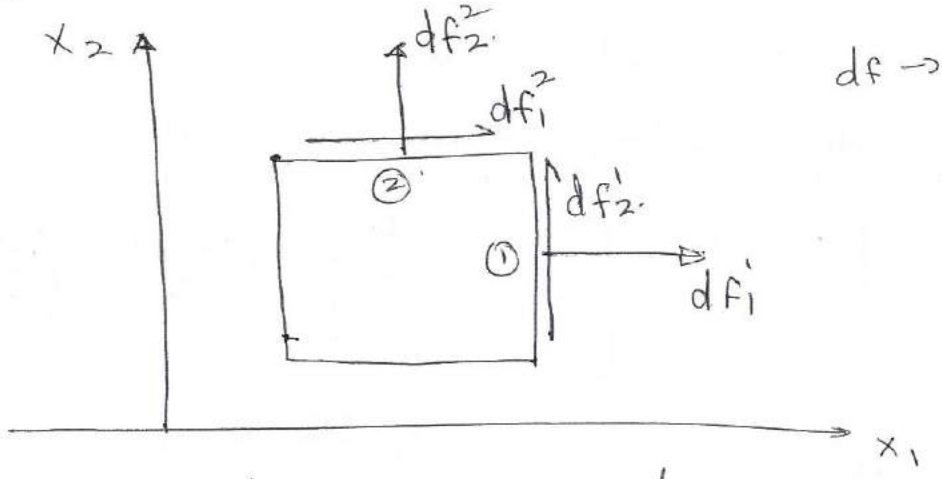
The Cauchy stress tensor, therefore, is

$$\sigma_{ij} = \begin{bmatrix} 0 & \mu\kappa & 0 \\ \mu\kappa & -\mu\kappa^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mu\kappa \begin{bmatrix} 0 & 1 & 0 \\ 1 & -\kappa & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The negative $\sigma_{22}$ seems ~~cond~~ counter-intuitive, but note that it is inherited directly from the $e_{22}$ strain component.

What can we say about the forces?

Let's assume that the deformed area is a cube.




$df \rightarrow$ Cauchy forces on the deformed areas.

For Cauchy stress, in general

$$t_i = \sigma_{ji} n_j = \frac{df_i}{dA}$$

On face ①: $\hat{n} = \{1, 0, 0\}$.

$$\text{i.e. } \frac{df_i'}{dA} = \sigma_{1i} n_1 + \sigma_{2i} n_2 + \sigma_{3i} n_3 = \sigma_{1i}$$

$$\therefore \frac{df_1'}{dA} = \sigma_{11} = 0 \quad , \quad \frac{df_2'}{dA} = \sigma_{12} = \mu\kappa \quad , \quad \frac{df_3'}{dA} = \sigma_{13} = 0$$

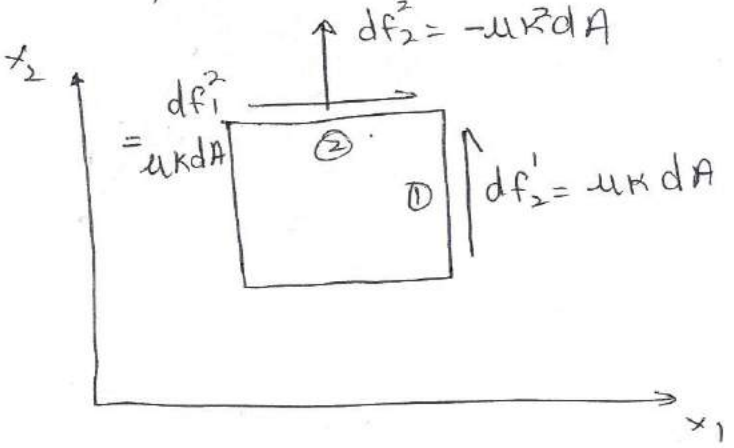
$$\text{i.e. } df_1' = 0 \quad , \quad \underline{df_2' = \mu\kappa dA} \quad , \quad df_3' = 0.$$

On face (2): $\hat{n} = \{0, 1, 0\}$, so the tractions are

$$\frac{df_1^2}{dA} = \mu K, \quad \frac{df_2^2}{dA} = -\mu K^2, \quad \frac{df_3^2}{dA} = 0.$$

i.e. $df_1^2 = \mu K\, dA, \quad df_2^2 = -\mu K^2 dA, \quad df_3^2 = 0.$

i.e.




So, we can physically interpret the system of forces as the forces $df_2'$ & $df_1^2$ applying a shear & causing a change in angle, while force $df_2^2$ on face (2) is a compressive force needed to keep the 2-face from deflecting in the 2-direction.

B] Lagrangian stress

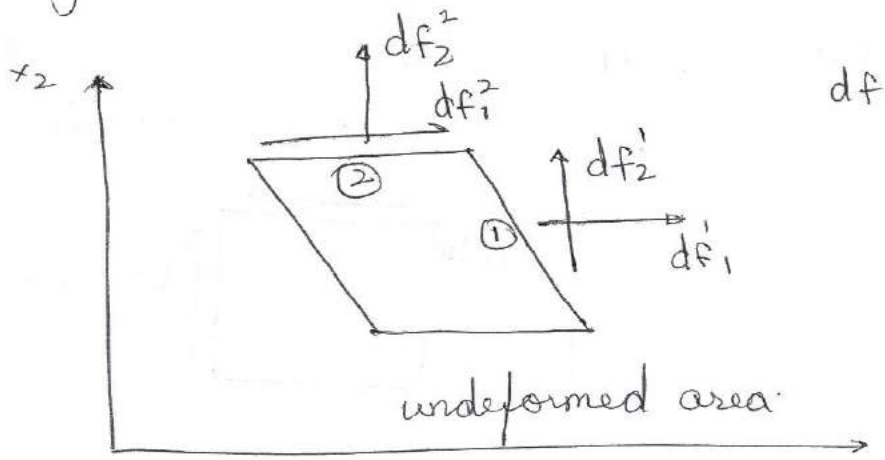
$$\sigma_{Pi}^L = \frac{\rho^0}{\rho} \sigma_{ij} \frac{\partial x_P^0}{\partial x_j} = \sigma_{i1} \frac{\partial x_P^0}{\partial x_1} + \sigma_{i2} \frac{\partial x_P^0}{\partial x_2}.$$

Then, doing the maths, we get

$$\sigma_{Pi}^L = \mu K \begin{bmatrix} -K & 1+K^2 & 0 \\ 1 & -K & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Note that $\sigma^L_{pi}$ is not symmetric. Also, in contrast with $\sigma_{ij}$, it has a non-zero $\sigma^L_{11}$ component.

Let's try to reason what the forces are like.




df → Lagrange forces

Note: The undeformed area will look like the above since we assumed that the deformed area is a cube.

The Lagrangian traction vectors are

$$t_i^L = \sigma^L_{pi}\, n^0_p = \frac{df_i}{dA^0}$$

To use this, we first need to find the orientations of the faces ($dA^0$). Recall, that deformed & undeformed areas are related by

$$n^0_p\, dA^0 = \frac{\rho}{\rho^0}\frac{\partial x_i}{\partial x^0_p}\, n_i\, dA$$

On face ①: $\hat{n} = \{1, 0, 0\}$

$$\therefore \quad n^0_p\, dA^0 = \frac{\partial x_1}{\partial x^0_p}\, dA = \{1, k, 0\}\, dA$$

i.e. the direction & magnitude of the undeformed area (face-1) are:

$$n_p^0 = \left\{ \frac{1}{\sqrt{1+K^2}}, \frac{K}{\sqrt{1+K^2}}, 0 \right\} \text{ and } dA^0 = \sqrt{1+K^2}\, dA$$

For face (2): $\hat{n} = \{0, 1, 0\}$. So, we obtain

$$n_p^0 = \{0, 1, 0\} \quad \& \quad dA^0 = dA$$

Now that we know the areas, we can calculate the forces.

On face 1: $\dfrac{df_i'}{dA^0} = \sigma_{1i}^L n_1^0 + \sigma_{2i}^L n_2^0 + \sigma_{3i}^L n_3^0$

i.e. $$\frac{df_1'}{dA^0} = -uK^2 \frac{1}{\sqrt{1+K^2}} - uK \frac{K}{\sqrt{1+K^2}} + 0 = 0$$

$$\frac{df_2'}{dA^0} = u(1+K^2)\frac{K}{\sqrt{1+K^2}} - uK^2 \frac{K}{\sqrt{1+K^2}} + 0$$

$$= \frac{uK}{\sqrt{1+K^2}}$$

Thus, the forces are:

$$df_1' = 0$$

$$df_2' = \frac{uK}{\sqrt{1+K^2}} dA^0$$

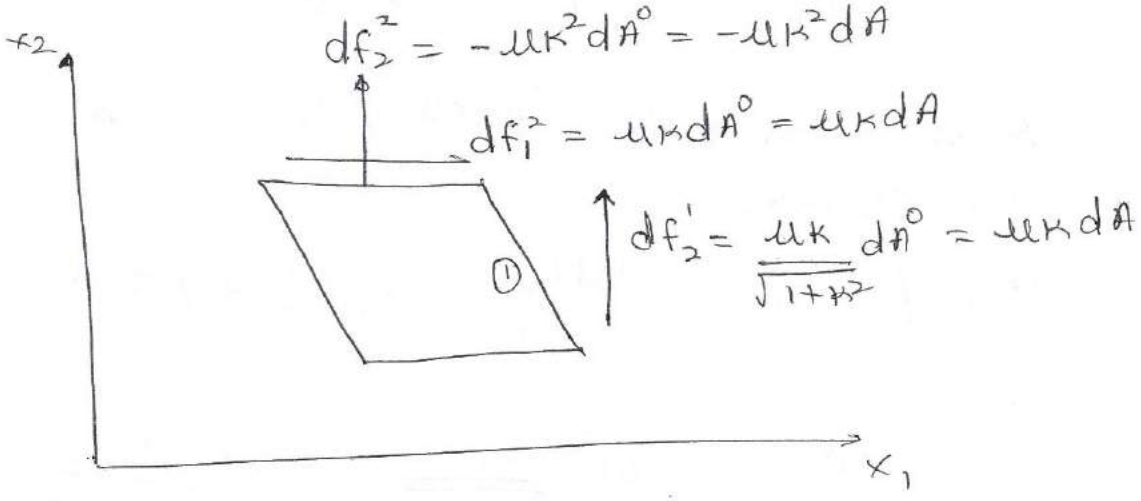
$$= \frac{uK}{\sqrt{1+K^2}} \sqrt{1+K^2} \cdot dA = uK\, dA$$

Similarly, on face 2:

$$\frac{df_1^2}{dA^0} = \mu K \quad \text{or} \quad df_1^2 = \mu K dA^0 = \mu K dA$$

$$\frac{df_2^2}{dA^0} = -\mu K^2 \quad \text{or} \quad df_2^2 = -\mu K^2 dA^0 = -\mu K^2 dA$$

Thus, the force system is




The forces on both faces are the same as for Cauchy stress (as expected). Note that the forces are with respects to coordinate directions and not with respect to the rotated face.

c] Kirchoff stresses:

$$\sigma_{pq}^{K} = \frac{\rho^0}{\rho}\,\sigma_{ij}\,\frac{\partial x_p^0}{\partial x_i}\,\frac{\partial x_q^0}{\partial x_j}$$

$$= \sigma_{12}\left[\frac{\partial x_p^0}{\partial x_1}\frac{\partial x_q^0}{\partial x_2} + \frac{\partial x_p^0}{\partial x_2}\frac{\partial x_q^0}{\partial x_1}\right] + \sigma_{22}\left[\frac{\partial x_p^0}{\partial x_2}\frac{\partial x_q^0}{\partial x_2}\right]$$

solving, we get

$$\sigma_{pq}^{K} = \mu\kappa\begin{bmatrix} -2\kappa-\kappa^3 & 1+\kappa^2 & 0 \\ 1+\kappa^2 & -\kappa & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Note that the magnitude of shear deformation is governed by the parameter $\kappa$. If $\kappa$ is small all 3 stress tensors approach the same value.

Also, observe that the simple constitutive relation $\sigma_{ij} = 2\mu e_{ij}$ in the Eulerian variables does not lead to an analogous simple relation between $\sigma_{ij}^{K}$ and $E_{ij}$.

Lecture 13.

Simple shear

Deformation :- $x_1 = x_1^0 + k x_2^0$

$x_2 = x_2^0$

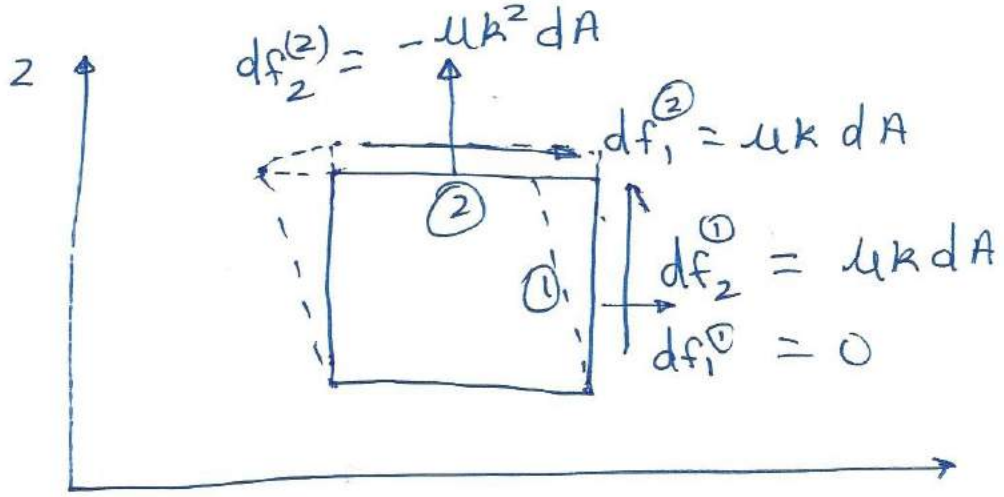
$x_3 = x_3^0$

Assume :- $\sigma_{ij} = 2\mu e_{ij}$

I] Cauchy stress :-

$$2e_{ij} = \delta_{ij} - \frac{\partial x_p^0}{\partial x_i}\frac{\partial x_p^0}{\partial x_j} = \begin{bmatrix} 0 & k & 0 \\ k & -k^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

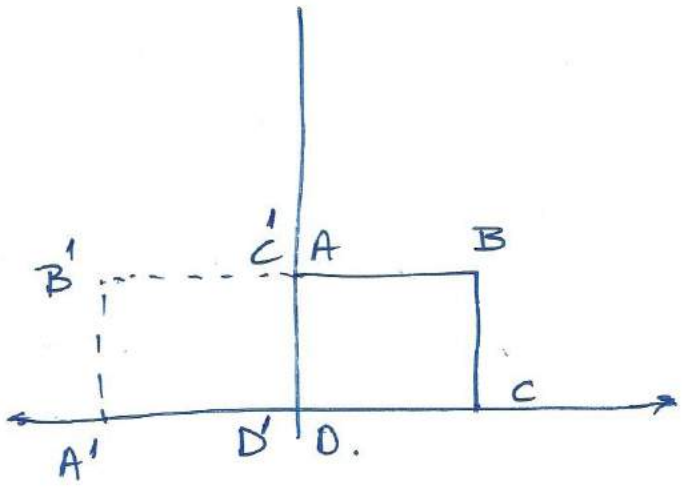
$$\sigma_{ij} = \begin{bmatrix} 0 & \mu k & 0 \\ \mu k & -\mu k^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$




Cauchy forces on the deformed area.

$$\rightarrow \quad \sigma_{ij} = \begin{bmatrix} 50 & 20 & 0 \\ 20 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$t_i = \sigma_{ji} n_j$$




For surface $A'B'$

$$\hat{n} = \{-1, 0, 0\}$$

$$t_1 = \sigma_{11} n_1 + \sigma_{21} n_2 + \sigma_{31} n_3$$

$$t_1 = -50 + 0 + 0.$$

$$t_1 = -50$$

$$t_2 = \sigma_{12} n_1 + \sigma_{22} n_2 + \sigma_{32} n_3$$

$$t_2 = -20$$

$$t_3 = 0$$

} (1)

For surface $B'C'$ :-

$$\hat{n} = \{0, 1, 0\}$$

$$t_1 = \sigma_{11} n_1 + \sigma_{21} n_2 + \sigma_{31} n_3 = 20$$

$$t_2 = 0$$

$$t_3 = 0$$

} (2)

II] Lagrange stresses:

$$\sigma_{pi}^{L} = \frac{\rho^{\circ}}{\rho} \sigma_{ij} \frac{\partial x_p^{\circ}}{\partial x_j}$$

$$\because \quad \sigma_{ij} = 0 \quad \forall \quad i,j = 3$$

$$\sigma_{pi}^{L} = \sigma_{i1} \frac{\partial x_p^{\circ}}{\partial x_1} + \sigma_{i2} \frac{\partial x_p^{\circ}}{\partial x_2}$$

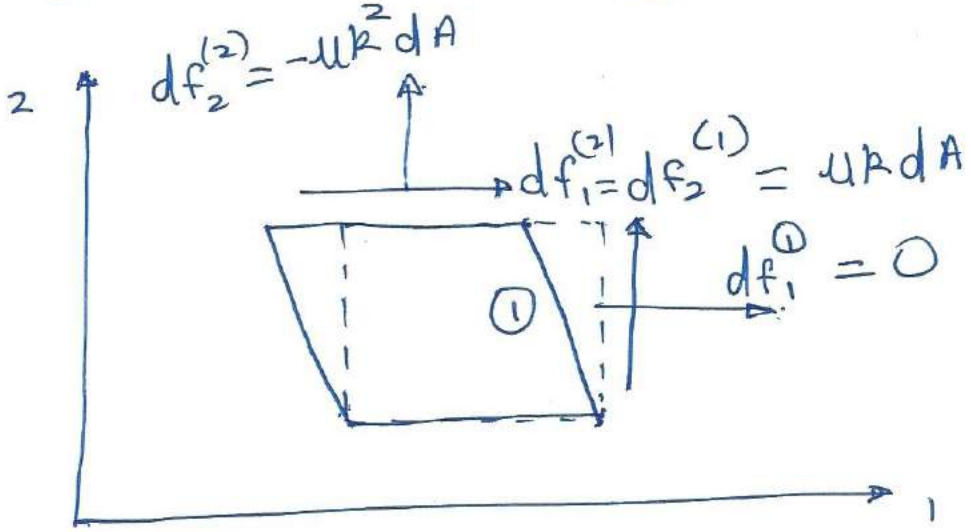
Ex :-

$$\sigma_{11}^{L} = \sigma_{11} \frac{\partial x_1^{\circ}}{\partial x_1} + \sigma_{12} \frac{\partial x_1^{\circ}}{\partial x_2} \qquad x_1 = x_1^{\circ} + kx_2^{\circ}$$

$$= 0 + \mu k (-k)$$

$$\sigma_{11}^{L} = -\mu k^2$$

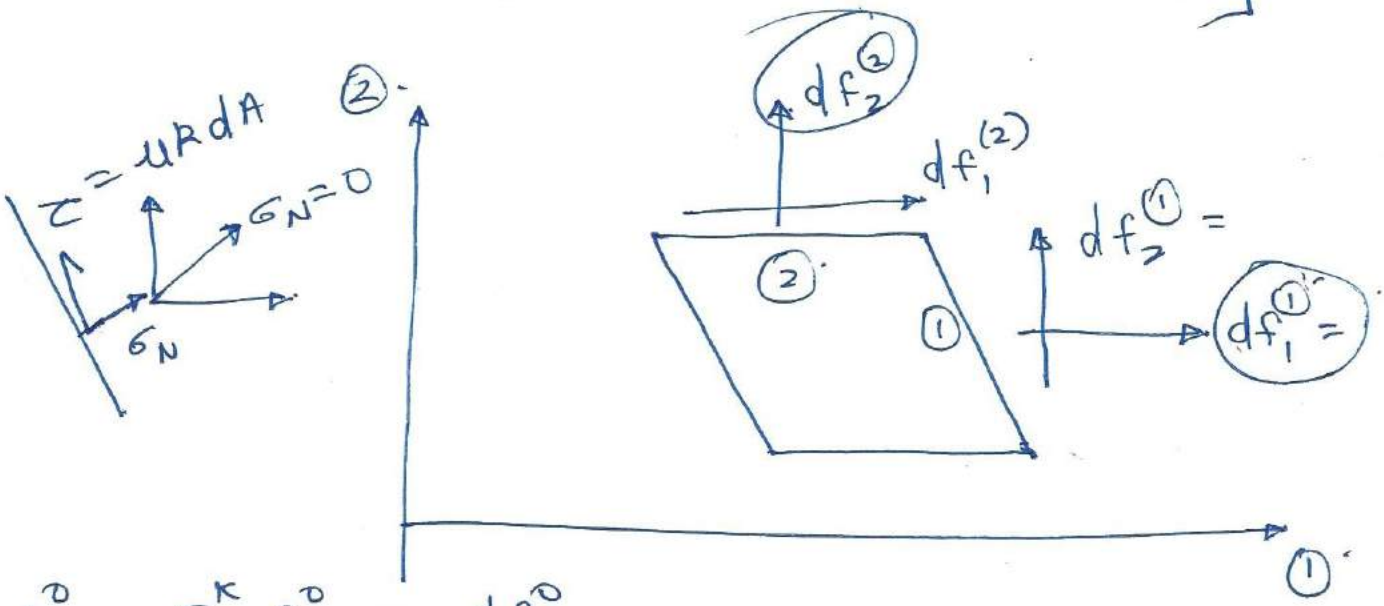
$$\sigma_{pi}^{L} = \begin{bmatrix} -\mu k^2 & \mu k + \mu k^3 & 0 \\ \mu k & -\mu k^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$



III] Kirchoff stress

$$\sigma^{K}_{pq} = \frac{\rho^{\circ}}{\rho}\,\sigma_{ij}\,\frac{\partial x^{\circ}_{p}}{\partial x_i}\,\frac{\partial x^{\circ}_{q}}{\partial x_j}$$

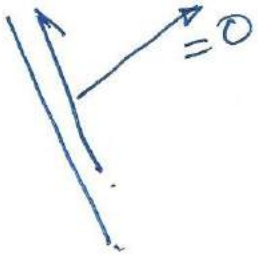
$$= \mu k \begin{bmatrix} -2k - k^3 & 1 + k^2 & 0 \\ 1 + k^2 & -k & 0 \\ 0 & 0 & 0 \end{bmatrix}$$




$$t^{\circ}_{i} = \sigma^{K}_{pi}\, n^{\circ}_{p} = \frac{df^{\circ}_{i}}{dA^{\circ}}$$

where : $df^{\circ}_{i} = \frac{\partial x^{\circ}_{i}}{\partial x_j}\, df_i$

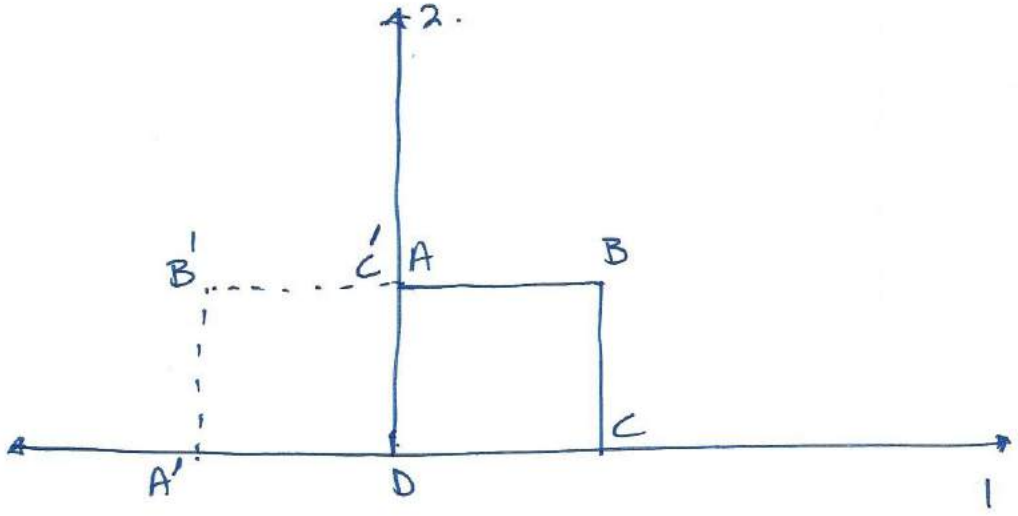
$$n^{\circ}_{p}\, dA^{\circ} = \frac{\rho}{\rho^{\circ}}\,\frac{\partial x_i}{\partial x^{\circ}_{p}}\, n_i\, dA$$




$$df'_1 = df^{2}_{2} = -\mu k^2\, dA$$

$$df'_2 = \mu k\, dA \quad , \quad df^{2}_{1} = \mu k (1 + k^2)\, dA$$

Example:




- Rigid body rotated by $\frac{\pi}{2}$

- Assume :- $\sigma_{ij} = \begin{bmatrix} 50 & 20 & 0 \\ 20 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ , $dA^\circ = 1$

- Deformation :-

$$x_1 = -x_2^\circ \;,\; x_2 = x_1^\circ \;,\; x_3 = x_3^\circ$$

i.e. $$x_1^\circ = x_2 \;,\; x_2^\circ = -x_1 \;,\; x_3^\circ = x_3$$

Q] Find $\sigma_{ij}^L$ , $\sigma_{ij}^K$ using their definitions.

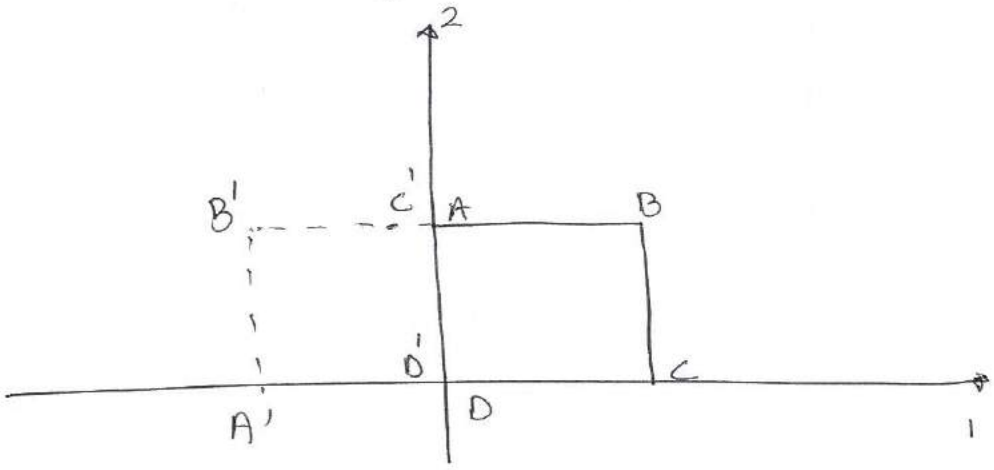
i.e. For Lagrange stress tensor :-

$$t_i^\circ = \sigma_{ji}^L n_j^\circ \;,\; t_i^\circ = \frac{df_i^\circ}{dA^\circ} = \frac{df_i}{dA^\circ}$$

Kirchoff :- $t_i^\circ = \sigma_{ji}^K n_j^\circ$ , $t_i^\circ = \frac{df^\circ}{dA^\circ}$ ; $df_i^\circ = \frac{\partial x_i^\circ}{\partial x_j} df_j$

Example :-

- Consider the case of a rigid body rotated by $\pi/2$.




Deformation:

$$x_1 = -x_2^0$$
$$x_2 = x_1^0$$
$$x_3 = x_3^0$$

Or

$$x_1^0 = x_2$$
$$x_2^0 = -x_1$$
$$x_3^0 = x_3$$

Assume:

$$\sigma_{ij} = \begin{bmatrix} 50 & 20 & 0 \\ 20 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad dA^0 = 1$$

Find: Lagrangian & Kirchoff stress tensors using their definitions.

i.e. $t_i^0 = \sigma_{ji}^L n_j^0$, where $t_i^0 = \dfrac{df_i^0}{dA^0} = \dfrac{df_i}{dA^0}$

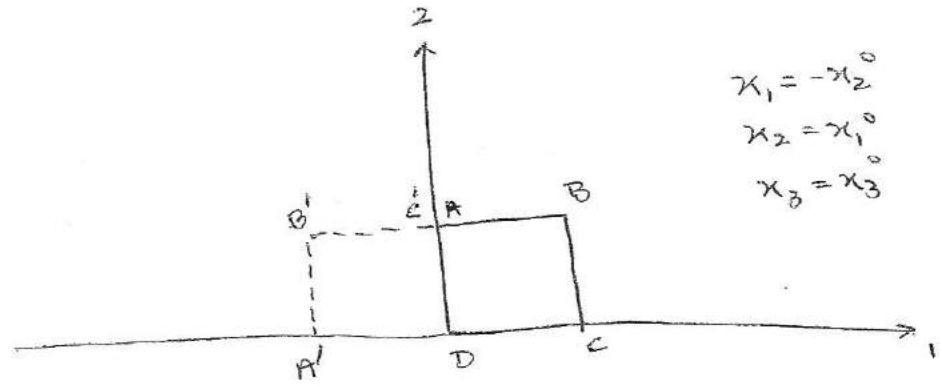
$\sigma_{ji}^L \rightarrow$ Lagrangian stress tensor

& $t_i^0 = \sigma_{ji}^K n_j^0$, where $t_i^0 = \dfrac{df_i^0}{dA^0}$

$$df_i^0 = \frac{\partial x_i^0}{\partial x_j} df_j$$

~~① Find tractions for Cauchy.~~

3·14




$x_1 = -x_2^0$, $x_2 = x_1^0$, $x_3 = x_3^0$ ; $x_1^0 = x_2$, $x_2^0 = -x_1$, $x_3^0 = x_3$

We are given:

$$[\sigma_{ij}] = \begin{bmatrix} 50 & 20 & 0 \\ 20 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

We have the relationship between the Cauchy stress ~~tensor~~ vector and traction vectors.

$$t_i = \sigma_{ji} n_j = \frac{df_i}{dA}$$

For the surface $A'B'$ : $\hat{n} = (-1, 0, 0)$ → After deformation

$$t_1 = \sigma_{11} n_1 + \sigma_{21} n_2 + \sigma_{31} n_3$$
$$t_1 = 50(-1) = -50$$
$$t_2 = \sigma_{12} n_1 + \sigma_{22} n_2 + \sigma_{32} n_3$$
$$t_2 = 20(-1) = -20$$
$$t_3 = 0. \qquad \} \ (1)$$

Similarly for the surface $B'C'$ : $\hat{n} = (0, 1, 0)$ → After deformation

we get

$$t_1 = \sigma_{11} n_1 + \sigma_{21} n_2 + \sigma_{31} n_3 = 20$$
$$t_2 = 0$$
$$t_3 = 0 \qquad \} \ (2)$$

⇒ Lagrangian stress:

For Lagrangian stress, the resultant force $df^\circ$ referred to the undeformed configuration is assumed to be identical to the force $df$ acting on the deformed area.

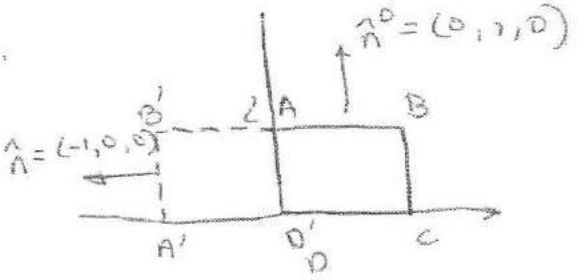
i.e. $df_i^\circ = df_i$

The Lagrangian stress tensor is defined as $t_i^\circ = \sigma_{ji}^L n_j^\circ$, where $t_i^\circ = \frac{df_i^\circ}{dA^\circ} = \frac{df_i}{dA^\circ}$

★ For the given problem, since it is a rigid body rotation, $dA^\circ = dA$ and we can see that $dA^\circ = 1$

Again, for the surface AB, we look at it in its undeformed configuration.

We have $\hat{n}_i^\circ = (0, 1, 0)$.




From ★, we can write

$df_i = t_i \, dA = (df_1, df_2, df_3)$

i.e. $df_1 = t_1 = df_1^\circ = -50$

$df_2 = t_2 = df_2^\circ = -20$

$df_3 = t_3 = df_3^\circ = 0$

∴ we can write :-

$$t_1^0 = \sigma_{11}^L n_1^0 + \sigma_{21}^L n_2^0 + \sigma_{31}^L n_3^0$$

i.e. $-50 = 0 + \sigma_{21}^L n_2^0 + 0$

i.e. $\underline{\sigma_{21}^L = -50}$

similarly,

$$t_2^0 = \sigma_{12}^L n_1^0 + \sigma_{22}^L n_2^0 + \sigma_{32}^L n_3^0$$

i.e. $-20 = \sigma_{22}^L (1)$

i.e $\underline{\sigma_{22}^L = -20}$.

we can obtain $\sigma_{11}^L$ and $\sigma_{12}^L$ from surface BC. Note that the Lagrangian stress tensor is not symmetric, so $\sigma_{12}^L \neq \sigma_{21}^L$.

For BC, before deformation, we have,

$\hat{n} = (1, 0, 0)$

$$t_1^0 = \sigma_{11}^L n_1^0 + \sigma_{21}^L n_2^0 + \sigma_{31}^L n_3^0 \qquad \left( t_1^0 = df_1 \cdot dA_0 = 20 \right)$$

i.e. $20 = \sigma_{11}^L (1) + 0 + 0$

i.e. $\sigma_{11}^L = 20$.

similarly,

$$t_2^0 = \sigma_{12}^L n_1^0 + \sigma_{22}^L n_2^0 + \sigma_{32}^L n_3^0$$

i.e. $0 = \sigma_{12}^L (1) + 0 + 0$

i.e. $\sigma_{12}^L = 0$

Thus we can write the final form as

$$\sigma_{ij}^{L} = \begin{bmatrix} 20 & 0 & 0 \\ -50 & -20 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

We can check this answer by obtaining $\sigma_{ij}^{L}$ by directly using $\sigma_{ij}$.

The relation between the two is :-

$$\sigma_{ij}^{L} = J^{0} \frac{\partial x_j^{0}}{\partial x_p^{0}} \sigma_{pi}$$

where $J^{0} = \det\left[\frac{\partial x_i}{\partial x_j^{0}}\right] = \det\left|\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right| = 1$

$$\therefore \sigma_{11}^{L} = \frac{\partial x_1^{0}}{\partial x_1}\sigma_{11} + \frac{\partial x_1^{0}}{\partial x_2}\sigma_{21} + \frac{\partial x_1^{0}}{\partial x_3}\sigma_{31}$$

$$\sigma_{11}^{L} = 20$$

Similarly, we can get the other components.

ao $$\sigma_{ij}^{L} = \begin{bmatrix} 20 & 0 & 0 \\ -50 & -20 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, both results are in agreement.

⇒ Kirchoff stress tensor:

For Kirchoff stress, the resultant force $de^{0}$ referred to the undeformed configuration is assumed to be given by a transformation

of the force $df$ acting on deformed area

i.e. $$df_i^0 = \frac{\partial x_i^0}{\partial x_j} df_j$$

Consider face AB : $\hat{n} = (-1, 0, 0)$

$\hat{n}_0 = (0, 1, 0)$.

we had : $t_1 = -50$ , $t_2 = -20$ , $t_3 = 0$

i.e. $df_i = \{-50, -20, 0\}$

$\therefore$ $$df_i^0 = \frac{\partial x_i^0}{\partial x_j} df_i$$

$\Rightarrow$ $$df_1^0 = \frac{\partial x_1^0}{\partial x_1} df_1 + \frac{\partial x_1^0}{\partial x_2} df_2$$

$$df_1^0 = 0 \quad -20$$

i.e. $df_1^0 = -20$.

Similarly : $$df_2^0 = \frac{\partial x_2^0}{\partial x_1} df_1 + \frac{\partial x_2^0}{\partial x_2} df_2$$

$$= (-1)(-50)$$

$$df_2^0 = 50$$

$\therefore$ $$t_1^0 = \frac{df_1^0}{dA^0} = -20$$

$$t_2^0 = \frac{df_2^0}{dA^0} = 50$$

By defn : $t_i^0 = \sigma_{ji}^K n_j^0$.

i.e. $t_1^0 = \sigma_{11}^K n_1^0 + \sigma_{21}^K n_2^0 + \sigma_{31}^K n_3^0$

i.e. $-20 = \sigma_{21}^K (1)$

i.e. $\underline{\sigma_{21}^K = -20}$.

$$t_2^0 = \sigma_{12}^K n_1^0 + \sigma_{22}^K n_2^0 + \sigma_{32}^K n_3^0$$

i.e. $\underline{\sigma_{22}^K = 50}$

We will obtain the other components by looking at face BC: $\hat{n} = (0, 1, 0)$, $\hat{n}_0 = (1, 0, 0)$

We had:-

$$t_1 = \sigma_{11} n_1 + \sigma_{21} n_2 + \sigma_{31} n_3$$

i.e. $t_1 = 20$

$$t_2 = \sigma_{12} n_1 + \sigma_{22} n_2 + \sigma_{32} n_3$$

i.e $t_2 = 0$

$$t_3 = 0.$$

$\therefore df_i = \{20, 0, 0\}$

$$\therefore df_i^0 = \frac{\partial x_i^0}{\partial x_j} df_j$$

i.e. $$df_1^0 = \frac{\partial x_1^0}{\partial x_1} df_1 + \frac{\partial x_1^0}{\partial x_2} df_2$$

$$df_1^0 = 0$$

similarly,

$$df_2^0 = -20.$$

Now,

$$t_i^0 = \sigma_{ji}^K n_j^0$$

This gives.

$$t_1^0 = \sigma_{11}^K n_1^0 + \sigma_{21}^K n_2^0 + \sigma_{31}^K n_3^0$$

$$0 = \sigma_{11}^K (n_1)$$

i.e. $\sigma_{11}^K = 0.$

Similarly:

$$t_2^0 = \sigma_{12}^K n_1^0 + \sigma_{22}^K n_2^0 + \sigma_{32}^K n_3^0$$

i.e. $\sigma_{12}^K = -20$.

Thus :- $\sigma_{ij}^K = \begin{bmatrix} 0 & -20 & 0 \\ -20 & 50 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

We verify this using the mathematical relationship:

$$\sigma_{ji}^K = J^0 \frac{\partial x_i^0}{\partial x_m} \frac{\partial x_j^0}{\partial x_n} \sigma_{mn}$$

$$\sigma_{11}^K = \frac{\partial x_1^0}{\partial x_1} \frac{\partial x_1^0}{\partial x_1} \sigma_{11} + \frac{\partial x_1^0}{\partial x_1} \frac{\partial x_1^0}{\partial x_2} \sigma_{12} + \frac{\partial x_1^0}{\partial x_2} \frac{\partial x_1^0}{\partial x_1} \sigma_{21} + \frac{\partial x_1^0}{\partial x_2} \frac{\partial x_1^0}{\partial x_2} \sigma_{22}.$$

$\sigma_{11}^K = 0$

Similarly: $\sigma_{12}^K = -20$, $\sigma_{21}^K = -20$, $\sigma_{22} = 50$

i.e. $\sigma_{ij}^K = \begin{bmatrix} 0 & -20 & 0 \\ -20 & 50 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Thus, both methods give the same answer.

Exam 1

Chapter 1 :- Cartesian Tensors.

## 1.1 Indical notation

$$A_i = \{A_1, A_2, A_3\}. \qquad i \to 1, 2, 3.$$

$$A_{ij} = \begin{matrix} A_{11}, A_{12}, A_{13} \\ A_{21}, A_{22}, A_{23} \\ A_{31}, A_{32}, A_{33} \end{matrix}$$

- Summation convention

$$x_i y_i = a \qquad i \to \text{dummy index.}$$

$$x_1 y_1 + x_2 y_2 + x_3 y_3 = a$$

$$A_{iiK} \Rightarrow A_{11K} + A_{22K} + A_{33K} \qquad \begin{matrix} i \to \text{dummy.} \\ K \to \text{free index} \end{matrix}$$

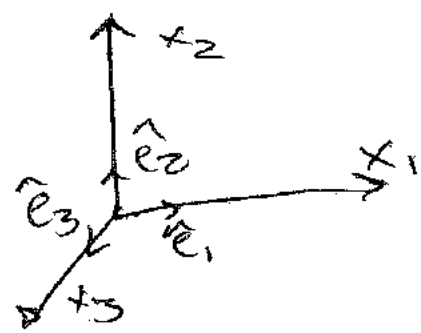
$A_{iij} B_{ij} \longrightarrow$ Wrong.

- Special symbols :-

① Base vectors :- unit direction vectors.

$$\hat{e}_1, \hat{e}_2, \hat{e}_3 \to \{x_1, x_2, x_3\}.$$

$$\hat{V} = V_1 \hat{e}_1 + V_2 \hat{e}_2 + V_3 \hat{e}_3$$
$$= V_i \hat{e}_i$$



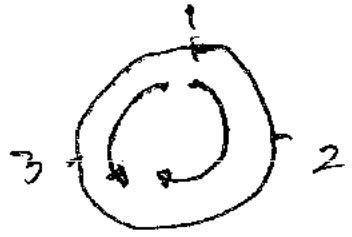
## (2) Kronecker delta : $\delta_{ij}$

$$\delta_{ij} = 1 \quad \text{if } i = j \quad \text{i.e. } [\delta_{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$= 0 \quad \text{if } i \neq j$$

- Properties :-
  - Symmetric : $\delta_{ij} = \delta_{ji}$
  - Trace : $\delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 3$
  - Contraction : $A_{iK}\,\delta_{ij} = A_{jK}$
  - Inner product : $\delta_{ij} = \hat{e}_i \cdot \hat{e}_j$

## (3) Permutation symbol :

$$\epsilon_{ijK} = \begin{cases} 1 & ijK \text{ form even permutation} \\ -1 & ijK \text{ form odd permutation} \\ 0 & \text{for all other cases.} \end{cases}$$




$\epsilon_{112} = 0$

$\epsilon_{123} = 1$

$\epsilon_{312} = 1$

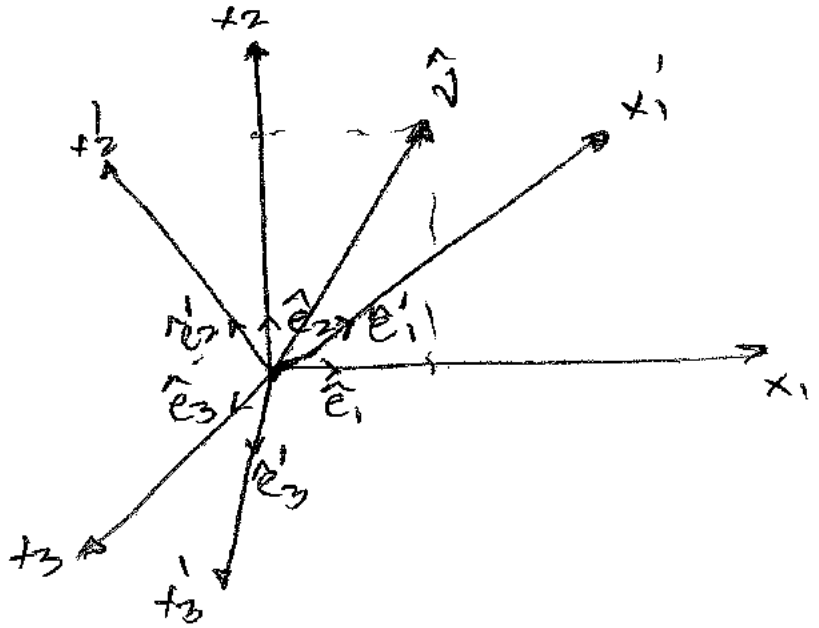
$\epsilon_{132} = -1$

- Properties :-
  - Anti symmetric → $\epsilon_{ijk} = -\epsilon_{ikj} = \epsilon_{kij}$
  - Trace → $\epsilon_{iik} = 0$
  - Contraction : $\epsilon_{ijs}\epsilon_{pqs} = \delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}$
  - determinant : $\det[a_{ij}] = \epsilon_{ijk} a_{1i} a_{2j} a_{3k}$
  - Cross Product : $\hat{e}_i \times \hat{e}_j = \epsilon_{ijk}\hat{e}_k$

★ $\hat{A}\cdot\hat{B} = A_i B_j \delta_{ij} = A_i B_i$

$\hat{A}\times\hat{B} = A_i B_j \epsilon_{ijk}\hat{e}_k$

## 1-2 Coordinate Transformation




$\hat{V} = V_i \hat{e}_i = V_i' \hat{e}_i'$

$V_i' = B_{ij} V_j$ ; $B_{ij} = \hat{e}_i' \cdot \hat{e}_j$ direction cosines.

$V_i = B_{ji} V_j'$

## 1.3 Scalar, Vectors, Tensor

- Scalar : $\phi(x_1, x_2, x_3) = \phi(x_1', x_2', x_3')$
- Vector : $\hat{v} = v_i \hat{e}_i = v_i' \hat{e}_i'$ 3-component

$$v_i' = \beta_{ij} v_j \quad \text{or} \quad [v'] = [\beta]\{v\}$$

- Tensor :

$$2^{nd} \text{ order} :- \quad t_{ij}' = \beta_{im} \beta_{jn} t_{mn}$$

$$3^{rd} \text{ order} : \quad t_{ijk}' = \beta_{im} \beta_{jn} \beta_{kp} t_{mnp}$$

- Isotropic tensor : $t_{ij}' = t_{ij}$

Eg : $\delta_{ij}$ , $\epsilon_{ijk}$

## 1.4 Second order tensors

- Symmetric : $S_{ij} = S_{ji}$ or $[S]^T = [S]$
- Anti symmetric : $A_{ij} = -A_{ji}$ or $[A]^T = -[A]$

★ Any $C_{ij}$ can be written as

$$C_{ij} = S_{ij} + A_{ij}$$

★ $S_{ij} A_{ij} = 0$

- Principal values of symmetric tensors.

$$V_i = T_{ij} n_j$$

$V_i \rightarrow$ Vector
$n_j \rightarrow$ direction
$T_{ij} \rightarrow 2^{nd}$ order sym.

There exists an $\hat{n}$, such that :-

$$V_i = \lambda n_i = T_{ij} n_j$$

i.e. $$[T_{ij} - \lambda \delta_{ij}] n_j = 0$$

$\lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)}$ → Eigen / Principal values
$\hat{n}^{(1)}, \hat{n}^{(2)}, \hat{n}^{(3)}$ → Eigen / Principal direct

↓
Always unit vectors.

---

## Chapter 2 :- Deformation

Deformation : A comparison of two states

$$x_i = x_i(x_1^0, x_2^0, x_3^0) \quad \text{or} \quad x_i^0 = x_i^0(x_1, x_2, x_3)$$

Lagrangian Eulerian.

Displacement : $\vec{u} = \underline{u_i = x_i - x_i^0}$

- Deformation gradient

$$\frac{\partial x_i}{\partial x_j^0} \quad / \quad \frac{\partial x_i^0}{\partial x_j}$$

$$\frac{\partial x_i}{\partial x_j^0} = \begin{matrix} 1\rightarrow \\ 2\rightarrow \\ 3\rightarrow \end{matrix} \begin{bmatrix} \frac{\partial x_1}{\partial x_1^0} & \frac{\partial x_1}{\partial x_2^0} & \frac{\partial x_1}{\partial x_3^0} \\ \frac{\partial x_2}{\partial x_1^0} & \frac{\partial x_2}{\partial x_2^0} & \frac{\partial x_2}{\partial x_3^0} \\ \frac{\partial x_3}{\partial x_1^0} & \frac{\partial x_3}{\partial x_2^0} & \frac{\partial x_3}{\partial x_3^0} \end{bmatrix}$$

$$dx_i = \frac{\partial x_i}{\partial x_j^0} \, dx_j^0$$

Jacobian : $J^0 = \det\left[\frac{\partial x_i}{\partial x_j^0}\right]$

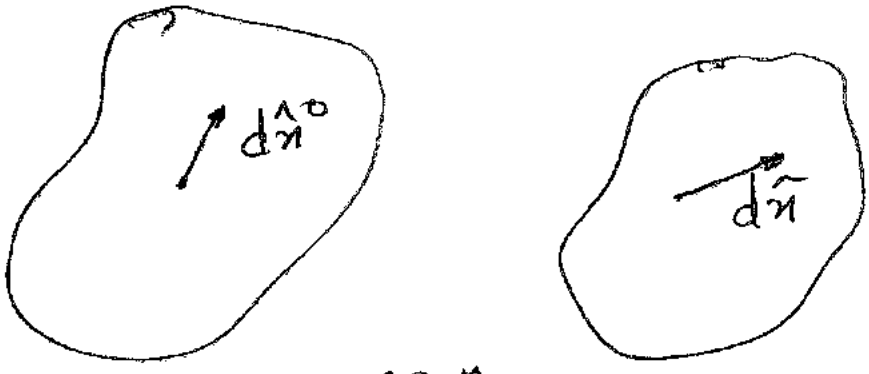
$J = \det\left[\frac{\partial x_i^0}{\partial x_j}\right]$

For valid deformation : $0 < J < \infty$

$0 < J^0 < \infty$.

## 2.2 Deformation of lines, areas, & volumes.

- Lines:




$$d\hat{x} = \frac{\partial x_i}{\partial x_j^0} d\hat{x}_j^0 \hat{e}_i$$

- Areas :- $dA_i = J^0 \frac{\partial x_K^0}{\partial x_i} dA_K^0$

- Volumes :- $dV = J^0 dV^0$

## 2.3 strains :-

- Elementary strains :-

(1) Engineering strain :- $e = \frac{ds - ds_0}{ds_0}$

(2) True strain $e^T = \frac{ds - ds_0}{ds}$

(3) Logarithmic strain: $e^N = \ln\left(\frac{ds}{ds_0}\right)$

- Tensorial strain measures:

- Lagrangian strain: $ds^2 - ds_0^2 = 2E_{ij}\, dx_i^0\, dx_j^0$

$$E_{ij} = \frac{1}{2}\left[\frac{\partial x_m}{\partial x_i^0}\frac{\partial x_m}{\partial x_j^0} - \delta_{ij}\right]$$

or $$2[E_{ij}] = \left[\frac{\partial x_m}{\partial x_i^0}\right]\left[\frac{\partial x_m}{\partial x_j^0}\right]^T - [I]$$

- Eulerian strain: $ds^2 - ds_0^2 = 2e_{ij}\, dx_i\, dx_j$

$$e_{ij} = \frac{1}{2}\left[\delta_{ij} - \frac{\partial x_m^0}{\partial x_i}\frac{\partial x_m^0}{\partial x_j}\right]$$

or $$2[e_{ij}] = [I] - \left[\frac{\partial x_m^0}{\partial x_i}\right]\left[\frac{\partial x_m^0}{\partial x_j}\right]^T$$

$E_{ij}$, $e_{ij}$ → 2nd order symmetric tensor

$$E_{ij} = e_{pq}\frac{\partial x_p}{\partial x_i^0}\frac{\partial x_q}{\partial x_j^0}$$

→ Principal values: There exists $\hat{n}$ such that:

$$[E_{ij} - \lambda\delta_{ij}]\, n_j^0 = 0$$

$\lambda^{(i)}$ → Principal strains

- Physical interpretation of strain components:

- Diagonal terms are a measure of "normal" strain

  → $E_{11}, E_{22}, E_{33}$ ; $e_{11}, e_{22}, e_{33}$

$\{ds_0, 0, 0\}$

$$E_1 = \frac{ds - ds_0}{ds_0}$$

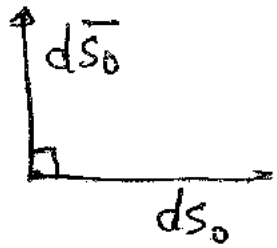
then :- $E_1 = \sqrt{1 + 2E_{11}} - 1$, $E_{11} = E_1 + \frac{1}{2}E_1^2$

$$e_1 = \frac{ds - ds_0}{ds}$$

$$e_1 = 1 - \sqrt{1 - 2e_{11}}$$

- Off-diagonal terms are a measure of shear strain

$$\sin \alpha_{12} = \frac{2E_{12}}{\sqrt{1+2E_{11}}\sqrt{1+2E_{22}}}$$




If $ds_0$ & $d\bar{s}_0$ are not initially 1$^{ar}$,

$$ds\, d\bar{s} \sin \alpha'_{12} = \left[2E_{KM} + \delta_{KM}\right] dx^0_K\, d\bar{x}^0_M$$

## 2.5 Strains in terms of displacement



$$u_i = x_i - x_i^\circ$$

$$E_{ij} = \frac{1}{2}\left[\frac{\partial u_i}{\partial x_j^\circ} + \frac{\partial u_j}{\partial x_i^\circ} + \frac{\partial u_m}{\partial x_i^\circ}\frac{\partial u_m}{\partial x_j^\circ}\right]$$

$$e_{ij} = \frac{1}{2}\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{\partial u_m}{\partial x_i}\frac{\partial u_m}{\partial x_j}\right]$$

- Infinitesimal strains

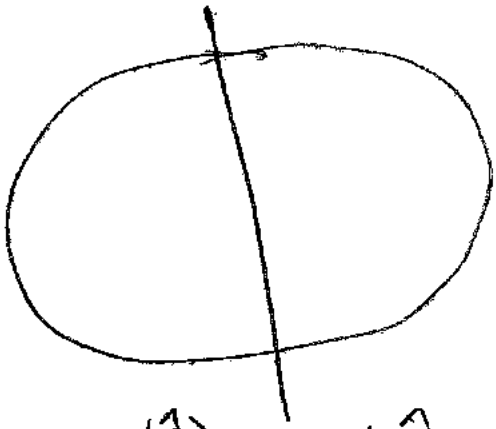
$$\left|\frac{\partial u_i}{\partial x_j^\circ}\right| \ll 1 \quad , \quad \left|\frac{\partial u_i}{\partial x_j}\right| \ll 1$$

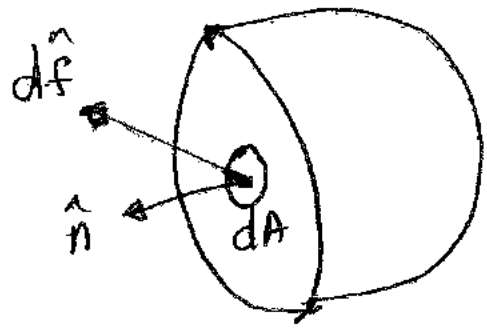
$$E_{ij} \simeq \bar{\epsilon}_{ij} = \frac{1}{2}\left[\frac{\partial u_i}{\partial x_j^\circ} + \frac{\partial u_j}{\partial x_i^\circ}\right]$$

$$e_{ij} \simeq \epsilon_{ij} = \frac{1}{2}\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right]$$

$$\bar{\epsilon}_{ij} \simeq \epsilon_{ij}$$

# Chapter 3: stress






Traction: $\hat{t}^{(\hat{n})} = \dfrac{d\hat{f}}{dA}$.

- Cauchy stress $\equiv$ Deformed area.

$\sigma_{ij}$ such that

$t_i = \sigma_{ji} n_j$ $\rightarrow$ $\sigma_{ij}$ is a 2nd order symm. tensor

- Equilibrium equations:-

$$\frac{\partial \sigma_{ij}}{\partial x_j} + \rho b_i = 0$$

$$\sigma_{ij} = \sigma_{ji}$$

- Principal stresses:-

$$[\sigma_{ij} - \sigma \delta_{ij}] n_j = 0$$

$\sigma_1, \sigma_2, \sigma_3 \rightarrow$ Principal stresses.

- Normal stress

$$\sigma_N = t_i n_i = \sigma_{ij} n_i n_j$$

- Shear stress

$$\tau^2 = t_i t_i - \sigma_N^2$$

## 3.3 stresses referred to undeformed state

### (1) Lagrangian stress tensor

If $df_i^0 = df_i$

then :- $$t_i^0 = \frac{df_i^0}{dA^0} = \frac{df_i}{dA^0} = \sigma_{ji}^L n_j^0$$

$$n_p^0 dA^0 = \int_{\rho^0} \frac{\partial x_i}{\partial x_p^0} n_i dA$$

$$\sigma_{pi}^L = \frac{\rho^0}{\rho} \sigma_{ji} \frac{\partial x_p^0}{\partial x_j} \Rightarrow \text{Not symmetric}$$

### (2) Kirchoff stress tensor

If $df_i^0 = \frac{\partial x_i^0}{\partial x_j} df_j$

then :- $$t_i^0 = \frac{df_i^0}{dA^0} = \frac{\partial x_i^0}{\partial x_j} \frac{df_j}{dA^0} = \sigma_{ji}^K n_j^0$$

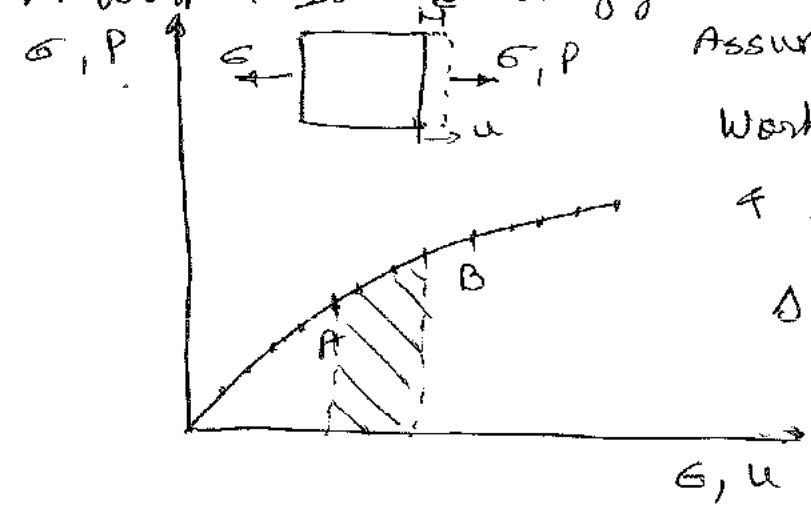
$$\sigma_{pq}^K = \frac{\rho^0}{\rho} \sigma_{ji} \frac{\partial x_p^0}{\partial x_i} \frac{\partial x_q^0}{\partial x_j}, \quad \sigma_{pq}^K = \sigma_{qp}^K$$

# Chapter 4: Elastic Materials

| Deformed Body | Undeformed body. |
|---|---|
| - Eulerian variables | Lagrangian variables |
| - Strain : $e_{ij} = \frac{1}{2}\left[\delta_{ij} - \frac{\partial x_m^\circ}{\partial x_i}\frac{\partial x_m^\circ}{\partial x_j}\right]$ | $E_{ij} = \frac{1}{2}\left[\frac{\partial x_m^\circ}{\partial x_i^\circ}\frac{\partial x_m}{\partial x_j^\circ} - \delta_{ij}\right]$ |
| - Stress : | |
| Cauchy : $\sigma_{ij}$ | Lagrangian : $\sigma_{ij}^L$ |
| $\Rightarrow t_i = \sigma_{ji} n_j = \frac{df_i}{dA}$ | $t_i^\circ = \sigma_{ji}^L n_j^\circ = \frac{df_i}{dA^\circ}$ |
| | Not symmetric. |
| | - Kirchoff : $\sigma_{ij}^K$ |
| | $t_i^\circ = \sigma_{ji}^K n_j^\circ = \frac{df_i^\circ}{dA^\circ}$ |

- How are stress & strain related?
  - Depends on the material.

## 4.1 Work & strain energy concepts.




Assume we keep P constant

Work done between state A & state B :

$$\Delta W = \int_A^B dW = \int_A^B \hat{P}\cdot d\hat{u} = \int_A^B P_i\, du_i$$

$$= \int_A^B \{P\}^T \{du\}.$$

★ $P_i$ & $u_i$ are generalized forces & displacements.

Assume virtual displacement in the body $du_i$.

Then external work done by $\hat{t}$ & $\hat{b}$

$$dW_e = \int_A t_i \, dA \cdot du_i + \int_V b_i \, dV \, du_i$$

$$= \int_A \sigma_{ji} n_j \, du_i \, dA + \int_V b_i \, du_i \, dV$$

By integral theorem:

$$dW_e = \int_V \frac{\partial}{\partial x_j} (\sigma_{ji} du_i) dV + \int_V b_i du_i dV$$

chain rule

$$= \int_V \sigma_{ji} \, d\left(\frac{\partial u_i}{\partial x_j}\right) dV + \int \left(\frac{\partial \sigma_{ji}}{\partial x_j} + b_i\right) du_i \, dV$$

$$= \int_V \sigma_{ji} d\left(\frac{\partial u_i}{\partial x_j}\right) dV$$

Equilibrium = 0 Eq

$$\frac{\partial u_i}{\partial x_j} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)$$

$$= \underset{\text{sym}}{\epsilon_{ji}} + \underset{\text{anti-sym}}{w_{ji}}$$

$$\therefore dW_e = \int_V \sigma_{ji}\, d(\epsilon_{ji} + \omega_{ji})\, dV$$

$$= \int_V \sigma_{ji}\, d\epsilon_{ji}\, dV + \int_V \sigma_{ji}\, \omega_{ji}\, dV$$

$\omega_{ji}$: (sym × anti-sym) = 0.

$$= \int_V \sigma_{ij}\, d\epsilon_{ij}\, dV$$

Work done will be stored as internal ~~stored~~ energy (strain energy).

Define: $d\bar{u} = \sigma_{ij}\, d\epsilon_{ij}$

$\bar{u}$ = strain energy density. (Energy / vol)

$$\therefore dW_e = \int_V d\bar{u}\, dV = dV \rightarrow$$ Product of Cauchy stress & small strain increm (ifinitesimal strain)

* In undeformed configuration, we get

$$dW_e = dV = \int_{V^0} \sigma^K_{PQ}\, dE_{PQ}\, dV^0.$$

∴ Cauchy stress / Eulerian (small) strain & Kirchoff stress / Lagrangian strain are energetically equivalent.

# 4.2 Elastic Constitutive Relations

- Material objectivity

Recall:- First order tensor: $\{\hat{v}'\} = [\hat{\beta}][\hat{v}]$

second order : $[\hat{T}'] = [\hat{\beta}][\hat{T}][\hat{\beta}]^T$

Objectivity ⟶ Frame indifferent

i.e. phenomena remains unchanged even if we change our reference frame.

To satisfy this, we need to formulate our constitutive relations using only quantities which are frame indifferent (objective).

Tensors which are frame-indifferent are called objective.

~~Consider~~ Can show that

Displacement: $\{\hat{u}'\} = [\hat{\beta}]\{\hat{u}\}$ Objective. ✓

Velocity : $\{\hat{v}'\} = [\hat{\beta}]\{\hat{u}\} + [\dot{\hat{\beta}}]\{\hat{u}\}$

Thus, not objective

Accl : Not objective

$\sigma_{ij}^K$ & $E_{ij}$ are objective. So we can use them.

- ~~Hyper~~ Elastic Materials:

How should we define Elastic materials?

Cauchy's approach

Definition 1: A material is elastic, if it possesses a homogeneous stress-free "natural" state, and if the stress depends only on deformation and not on the history of the deformation.

i.e. $\sigma_{ij}^{K} = \overset{\sigma_{ij}}{\cancel{D}}(E_{kl})$ Elastic material.

or $\sigma_{ij} = f_{ij}(E_{kl})$

Definition 2: The work done by the applied forces is transformed completely into strain (potential) energy, and this strain energy is completely recoverable.

i.e. the material system is conservative.

i.e. $\sigma_{ij}^{K} = \dfrac{\partial \bar{U}}{\partial E_{ij}}$ → Hyperelastic material.

This is ~~useful~~ useful for large deformations & for anisotropic materials.

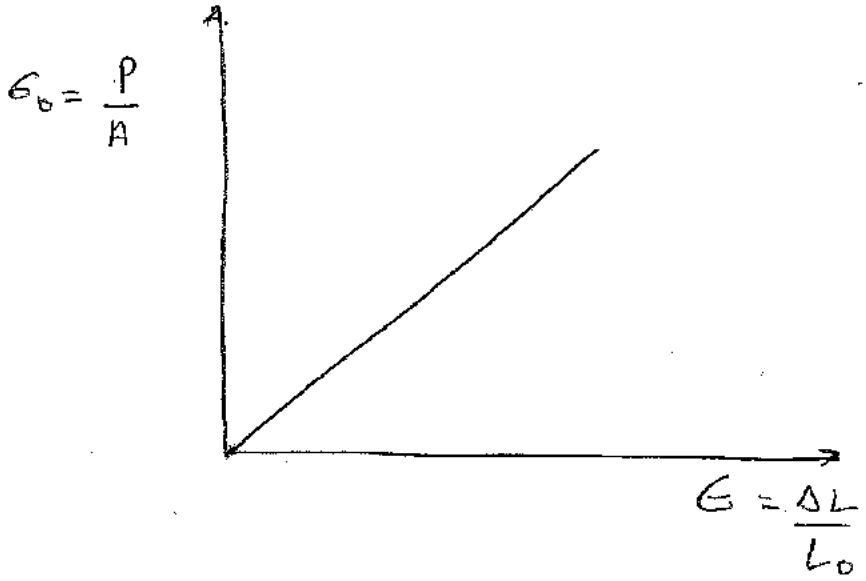
We can write:

$$\sigma_{ij}^{K} = \frac{\partial \bar{U}}{\partial E_{ij}} = 2\frac{\partial \bar{U}}{\partial C_{ij}} \quad ; C_{ij} = \sum_{m} \frac{\partial x_m}{\partial x_i^{\circ}} \frac{\partial x_m}{\partial x_j^{\circ}} = 2E_{ij} + \delta_{ij}$$

$C_{ij}$ → ~~Right~~ Cauchy-Green deformation tensor

Principal values of $C_{ij}$ are equivalent to principal stretches $\lambda_1, \lambda_2, \lambda_3$.

- Consider the large deformation testing of a uniaxial specimen.



$$\epsilon_T = \frac{\Delta W}{W_0}$$

We can show:

$$\lambda_1 = 1 + \epsilon \quad , \quad \lambda_2 = 1 + \epsilon_T = 1 - \upsilon\epsilon \quad , \quad \lambda_3 = 1 - \upsilon\epsilon$$

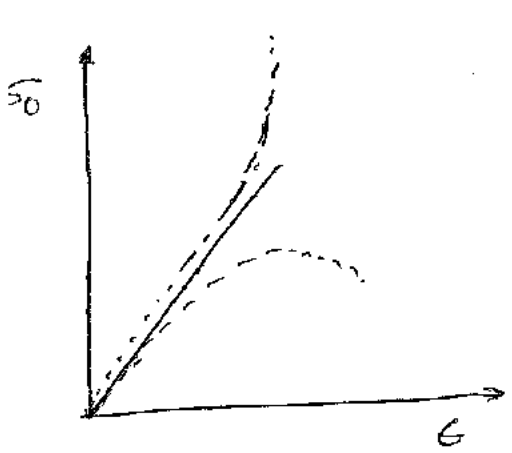
$$\upsilon = -\frac{\epsilon_T}{\epsilon}$$

Calculating strains, we get:

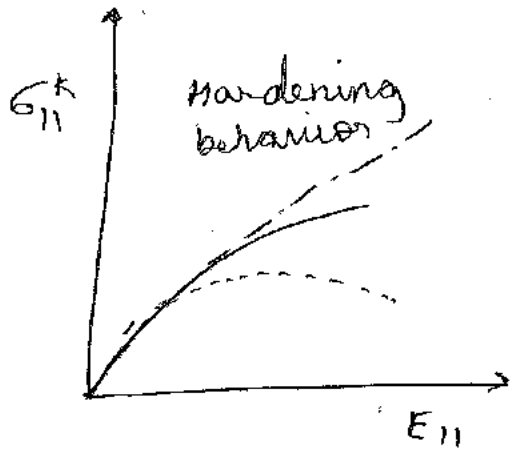
$$E_{11} = \epsilon + \frac{1}{2}\epsilon^2 \quad , \quad e_{11} = e - \frac{1}{2}e^2 = \frac{\epsilon}{(1+\epsilon)^2}\left[1 + \frac{\epsilon}{2}\right]$$

Stresses:

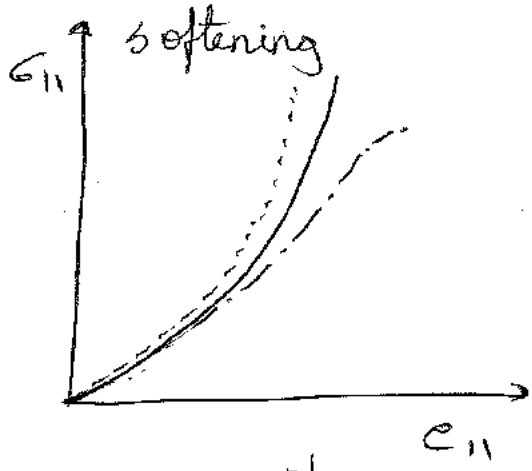
$$\sigma_{11}^{K} = \frac{1}{\lambda_1}\frac{dP}{dA_1^0} = \frac{\sigma_0}{1+\epsilon} \quad , \quad \sigma_{11} = \frac{1}{\lambda_2\lambda_3}\frac{dP}{dA_1^0} = \frac{\sigma_0}{(1-\upsilon\epsilon)^2}$$




recorded data




Kirchoff stress v/s Lagrangian strain




Cauchy stress v/s Eulerian strain.

Is the material physically linear or non-linear softening or hardening?

* Depends on which stress & strain measures you use!

---

- small strain Elastic materials

Assume: $\sigma_{ij} = f_{ij}(\epsilon_{kl})$

Taylor expansion gives:

$$\sigma_{ij} = f_{ij}(0) + \frac{\partial f_{ij}(0)}{\partial \epsilon_{kl}} \epsilon_{kl} + \frac{\partial^2 f_{ij}(0)}{\partial \epsilon_{kl} \partial \epsilon_{mn}} \epsilon_{kl} \epsilon_{mn} + \cdots$$

$f_{ij}(0)$ → Residual stress (does not depend on strain)

$$\sigma_{ij} = \frac{\partial f_{ij}(0)}{\partial \epsilon_{kl}} \epsilon_{kl} + \text{higher order terms.}$$

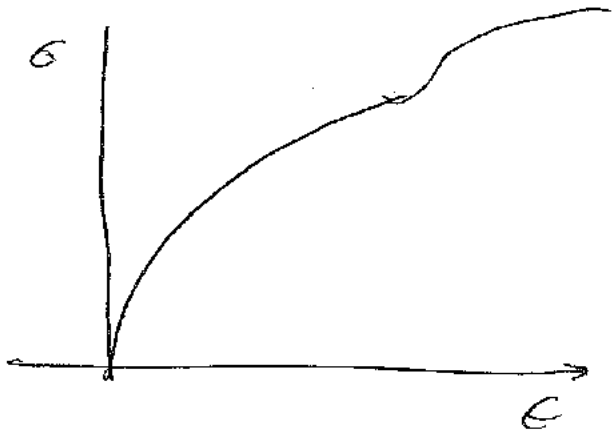
↓

Physical non-linearity

* Geometrical non-linearity:

Large deformation

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} - \frac{\partial u_m}{\partial x_i}\frac{\partial u_m}{\partial x_j}\right)$$




Linear Elasticity:

$$\sigma_{ij} = \frac{\partial f_{ij}(0)}{\partial \epsilon_{kl}} \epsilon_{kl} = C_{ijkl}\epsilon_{kl}$$

$C_{ijkl}$ : Elastic constants ; 81 constants.

$$C_{ijkl} = \frac{\partial f_{ij}}{\partial \epsilon_{kl}}$$

Recall: $\partial \bar{U} = \sigma_{ij}\, \partial \epsilon_{ij}$

$$\therefore \sigma_{ij} = \frac{\partial \bar{U}}{\partial \epsilon_{ij}} = C_{ijkl}\epsilon_{kl}$$

$$\therefore \frac{\partial C_{ijkl}\epsilon_{kl}}{\partial \epsilon_{kl}} = \frac{\partial^2 \bar{U}}{\partial \epsilon_{ij}\partial \epsilon_{kl}}$$

$dW_e \rightarrow$ External work done

$d\bar{u} \rightarrow$ strain energy density $= \sigma_{ij}\, d\epsilon_{ij}$

infinitesimal strain

$$dW_e = dU = \int_V \sigma_{ij}\, d\epsilon_{ij}\, dV = \int_{V^0} \sigma^K_{pq}\, dE_{pq}\, dV^0$$

Elastic Material:

(1) $\sigma^K_{ij} = G_{ij}(E_{KL})$ or $\sigma_{ij} = f_{ij}(\epsilon_{KL})$

(2) $\sigma^K_{ij} = \dfrac{\partial \bar{u}}{\partial E_{ij}} \rightarrow$ Used for hyperelastic materials

$$= 2\,\frac{\partial \bar{u}}{\partial C_{ij}}$$

$C_{ij} = 2E_{ij} + \delta_{ij} \rightarrow$ Right Cauchy-Green deformation tensor.

- Small strain elastic materials

Assume :- $\sigma_{ij} = f_{ij}(\epsilon_{kl})$

Taylor expansion gives

~~$\sigma_{ij} = f_{ij}(0)$~~

$$\sigma_{ij} = f_{ij}(\epsilon_{kl})\Big|_{\epsilon_{kl}=0} + \frac{\partial f_{ij}}{\partial \epsilon_{kl}}\Big|_{\epsilon_{kl}=0} \epsilon_{kl} + \frac{\partial^2 f_{ij}}{\partial \epsilon_{kl}\,\partial \epsilon_{mn}}\Big|_{\epsilon_{kl}=0,\ \epsilon_{mn}=0} \epsilon_{kl}\,\epsilon_{mn} + \cdots$$

↓

Residual stress → independent of strain.

$$\sigma_{ij} = \frac{\partial f_{ij}(0)}{\partial \epsilon_{kl}}\,\epsilon_{kl} + \text{higher order terms}$$

↓

Physical nonlinearity

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} + \frac{\partial u_m}{\partial x_i}\frac{\partial u_m}{\partial x_j}\right)$$

↓

Geometric non-linearity

Linear Elasticity

$$\sigma_{ij} = \frac{\partial f_{ij}(0)}{\partial \epsilon_{kl}}\,\epsilon_{kl} = C_{ijkl}\,\epsilon_{kl}$$

Generalized Hooke's law:

$$\sigma_{ij} = C_{ijkl}\,\epsilon_{kl}$$

$\downarrow$

Elastic constants $\rightarrow$ 81 constants

$$C_{ijkl} = \frac{\partial f_{ij}}{\partial \epsilon_{kl}}$$

Earlier: $\partial \bar{u} = \sigma_{ij}\,\partial \epsilon_{ij}$

$$\sigma_{ij} = \frac{\partial \bar{u}}{\partial \epsilon_{ij}} = C_{ijkl}\,\epsilon_{kl}$$

$$\frac{\partial^2 \bar{u}}{\partial \epsilon_{ij}\,\partial \epsilon_{kl}} = C_{ijkl} = \frac{\partial^2 \bar{u}}{\partial \epsilon_{kl}\,\partial \epsilon_{ij}}$$

$\Rightarrow \quad C_{ijkl} = C_{klij}$ i.e. 36 ~~cons~~ independent constants.

$\sigma_{ij} = \sigma_{ji} \longrightarrow \sigma_{12} = \sigma_{21}, \ldots$

$\epsilon_{ij} = \epsilon_{ji} \longrightarrow \epsilon_{12} = \epsilon_{21} \ldots$

# Voigt Notation / Engineering notation

Define : $\{\sigma\} = \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} = \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix}$

$$\{\epsilon\} = \begin{Bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ 2\epsilon_{23} = \gamma_{23} \\ 2\epsilon_{13} = \gamma_{13} \\ 2\epsilon_{12} = \gamma_{12} \end{Bmatrix} = \begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4, \gamma_1 \\ \epsilon_5, \gamma_2 \\ \epsilon_6, \gamma_3 \end{Bmatrix}$$

Engineering strain components

$\rightarrow$ $\sigma_{ij} = c_{ijkl}\,\epsilon_{kl}$

Can be reduced to

$\{\sigma\} = [c]\{\epsilon\}$

$\downarrow$

$6 \times 6$ matrix

$c$ $\rightarrow$ stiffness matrix

$S$ $\downarrow$ compliance matrix

$\{\epsilon\} = [S]\{\sigma\}$

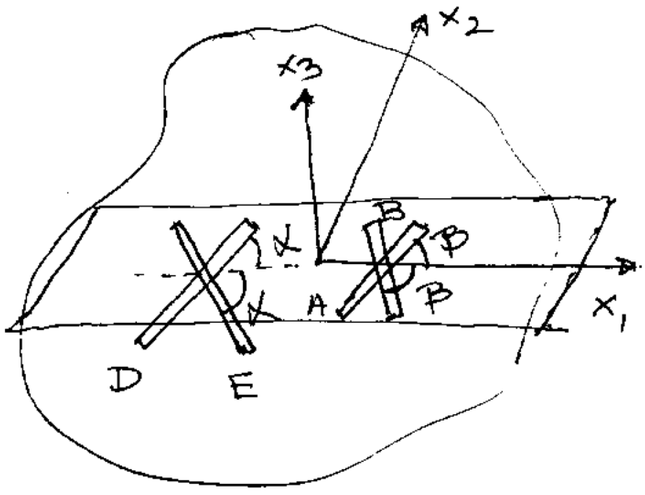
; $c_{ijkl} = c_{jilk}$

$\downarrow$

21 independent constants.

# 4.4 Elastic symmetries

## Plane of symmetry



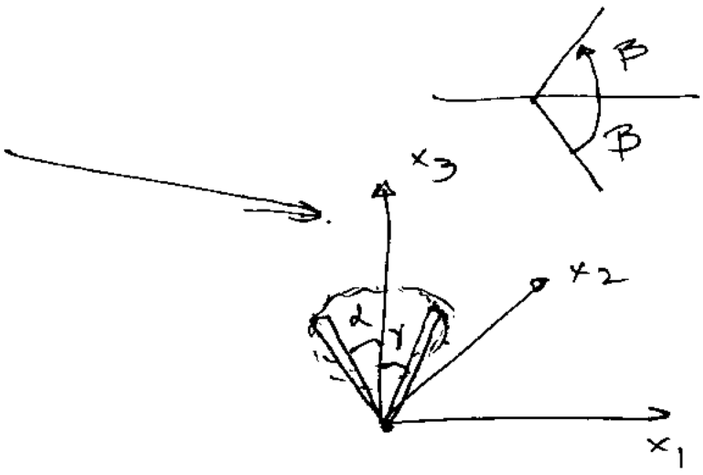

Element A :- $\sigma = C\epsilon$

B :- $\sigma = C\epsilon$

Element D : $\sigma = C'\epsilon$

E :⇉ $\sigma = C'\epsilon$

## Axis of symetry



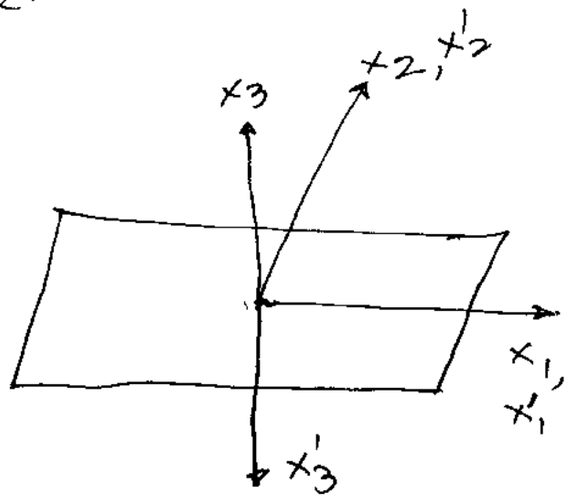

I⁄ Monoclinic system : One elastic symmetry plane.

Let $x_1, x_2, x_3$ be a co-ordinate system. Let the symmetric coordinate system be $x_1', x_2', x_3'$.

$\hat{e}_1' = \hat{e}_1 = \{1, 0, 0\}$

$\hat{e}_2' = \hat{e}_2 = \{0, 1, 0\}$

$\hat{e}_3' = -\hat{e}_3 = \{0, 0, -1\}$



$$B_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad \{\sigma\} = [C]\{\epsilon\} \ \star.$$

In the new co-ordinate system

$$\sigma'_{ij} = B_{ip} B_{jq} \sigma_{pq} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & -\sigma_{13} \\ \sigma_{21} & \sigma_{22} & -\sigma_{23} \\ -\sigma_{13} & -\sigma_{23} & \sigma_{33} \end{bmatrix} \quad (1)$$

$$\epsilon'_{ij} = B_{ip} B_{jq} \epsilon_{pq} = \begin{bmatrix} \epsilon_{11} & \epsilon_{12} & -\epsilon_{13} \\ \epsilon_{12} & \epsilon_{22} & -\epsilon_{23} \\ -\epsilon_{13} & -\epsilon_{23} & \epsilon_{33} \end{bmatrix} \quad (2)$$

Symmetry requires:

$$\{\sigma'\} = [C]\{\epsilon'\} \qquad -(3)$$

i.e. $$\sigma'_{11} = C_{11}\epsilon'_{11} + C_{12}\epsilon'_{22} + C_{13}\epsilon'_{33} + C_{14}2\epsilon'_{23} + C_{15}2\epsilon'_{13} + C_{16}2\epsilon'_{12}$$

From (1) & (2) into (3)

$$\sigma_{11} = C_{11}\epsilon_{11} + C_{12}\epsilon_{22} + C_{13}\epsilon_{33} - 2C_{14}\epsilon_{23} - 2C_{15}\epsilon_{13} + 2C_{16}\epsilon_{12} \qquad -(4)$$

But $\{\sigma\} = [C]\{\epsilon\}$

$$\sigma_{11} = C_{11}\epsilon_{11} + C_{12}\epsilon_{22} + C_{13}\epsilon_{33} + 2C_{14}\epsilon_{23} + 2C_{15}\epsilon_{13} + 2C_{16}\epsilon_{12} \qquad -(5)$$

i.e. $C_{14} = C_{15} = 0$

From other 5 equations:

$$C_{24} = C_{25} = C_{34} = C_{35} = C_{46} = C_{56} = 0$$

i.e. $[C] =$

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & C_{16} \\ C_{12} & C_{22} & C_{23} & 0 & 0 & C_{26} \\ C_{13} & C_{23} & C_{33} & 0 & 0 & C_{36} \\ 0 & 0 & 0 & C_{44} & C_{45} & 0 \\ 0 & 0 & 0 & C_{45} & C_{55} & 0 \\ C_{16} & C_{26} & C_{36} & 0 & 0 & C_{66} \end{bmatrix}$$

13 independent constants.

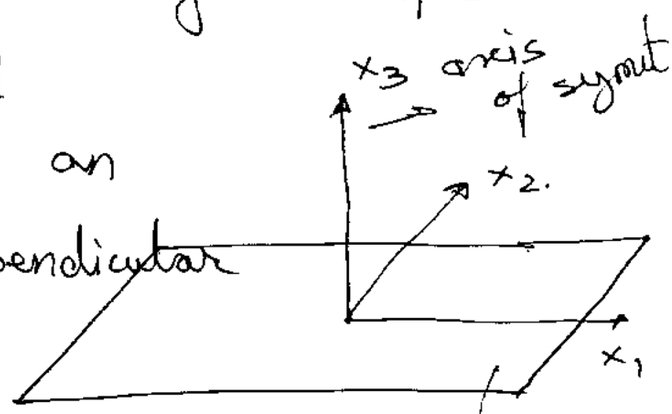
2] Orthotropic system: Three mutually perpendicular planes of symmetry.
i.e. no change in mechanical behaviour when $x_1, x_2, x_3$ are reversed.

$$[C] = \begin{bmatrix} C_{11} & C_{12} & C_{13} & & & \\ C_{12} & C_{22} & C_{23} & & 0 & \\ C_{13} & C_{23} & C_{33} & & & \\ & & & C_{44} & & \\ & 0 & & & C_{55} & \\ & & & & & C_{66} \end{bmatrix}$$

9 independent constants.

3) Hexagonal system: Tranversely isotropic

There exists a plane of symmetry in addition to an axis of symmetry perpendicular to the plane.




$$[C] = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{13} & 0 & 0 & 0 \\ C_{13} & C_{13} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(C_{11}-C_{12}) \end{bmatrix}$$

Plane of symmetry

5 - independent elastic constants.

4) Isotropic material:

Every plane is a plane of symmetry and every axis is an axis of symmetry. i.e. $[C]$ is independent of co-ordinate system.

$$[C] = \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{12} & C_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}(C_{11}-C_{12}) & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2}(C_{11}-C_{12}) & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(C_{11}-C_{12}) \end{bmatrix}$$

Another representation:

$$C_{11} = C_{22} = C_{33} = \lambda + 2\mu \qquad \lambda, \mu$$

$$C_{12} = C_{23} = C_{13} = \lambda \qquad \text{↳ Lamé constants.}$$

i.e. $$\sigma_{ij} = 2\mu\epsilon_{ij} + \lambda\epsilon_{kk}\delta_{ij}$$

$$2\mu\epsilon_{ij} = \sigma_{ij} - \frac{\lambda}{3\lambda+2\mu}\sigma_{kk}\delta_{ij}$$

Small strain Elastic material

$$\sigma_{ij} = f_{ij}(E_{kl})$$

Linear Elasticity : $\sigma_{ij} = \frac{\partial f_{ij}(0)}{\partial \epsilon_{kl}} \epsilon_{kl} = C_{ijkl}\, \epsilon_{kl}$

$C_{ijkl}$ → Elastic Constants

Voigt Notation :

$$\{\sigma\}^T = \{\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{23}, \sigma_{13}, \sigma_{12}\}$$

$$\{\epsilon\}^T = \{\epsilon_{11}, \epsilon_{22}, \epsilon_{33}, 2\epsilon_{23}, 2\epsilon_{13}, 2\epsilon_{12}\}$$

$\{\sigma\} = [C]\{\epsilon\}$ → Generalized Hooke's Law.

↳ 21 independent constants.

Different material systems :

- Monoclinic : 1 symmetry plane, 13 independent constants
- Orthotropic : 3 mutually orthogonal symmetry planes
  9 independent constants
- Hexagonal (Transversely isotropic)
  1 symmetry plane & 1 axis of symmetry
  5 independent constants.
- Isotropic : Direction independent,

$$\sigma_{ij} = 2\mu\epsilon_{ij} + \lambda \epsilon_{kk}\delta_{ij} \qquad \lambda, \mu$$

$$2\mu\epsilon_{ij} = \sigma_{ij} - \frac{\lambda}{3\lambda + 2\mu}\sigma_{kk}\delta_{ij} \qquad \text{"Lame" constants}$$

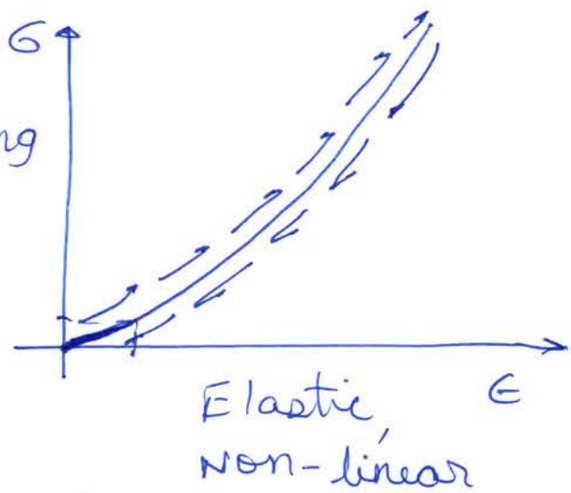
# - Linear Elastic Isotropic Materials

(1) Isotropic : $[C]$ independent of direction

(2) Elastic :

(i) Material returns to its original shape when loads are removed, loading & unloading paths are identical.

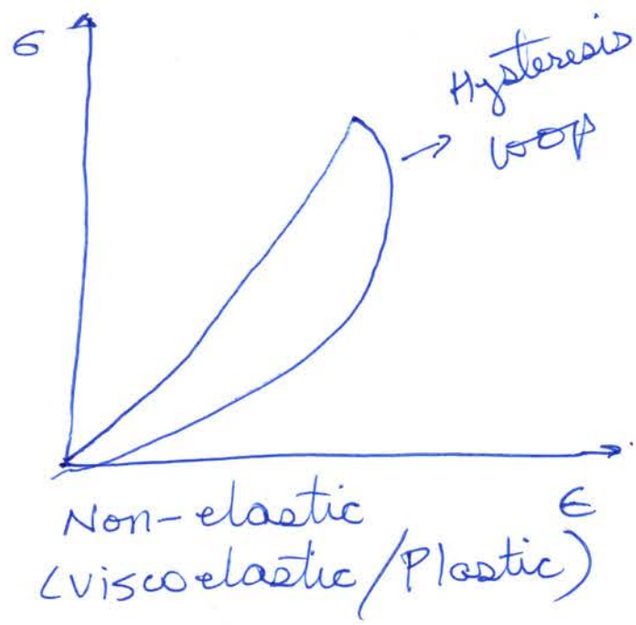
(ii) No dependence on rate of loading/straining




(3) Linear :

(i) stress is proportional to strain.

(ii) small strains & deformation.



- Commonly used elastic constant for LEIM.

(i) K - Bulk modulus - Measure of how compressible the material is.

(ii) E - Young's Modulus - Measure of stiffness i.e. How resistant the material is to deformation.

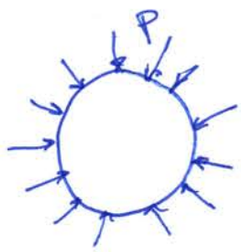
(iii) G - Shear Modulus - Measure of rigidity i.e. How resistant the material is to change of shape.

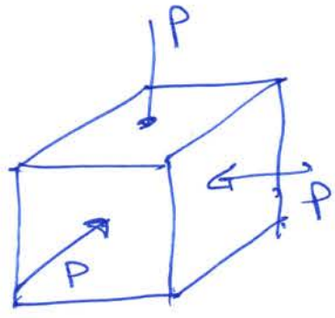
(iv) $\nu$ - Poisson's ratio - Measure of transverse contraction/elongation.

## I] Hydrostatic pressure:

$$\sigma_{ij} = -p\delta_{ij}$$

$p \rightarrow$ pressure




$$\therefore t_i = \sigma_{ij} n_j = -p\delta_{ij} n_j$$

$$t_i = -p n_i$$

i.e. traction vector ($\hat{t}$) is always $\perp^{ar}$ to any plane passing through a point.

Take contraction :-

$$\sigma_{ii} = -p\delta_{ii} = -3p$$

For isotropic material :-

$$\sigma_{ij} = 2\mu\epsilon_{ij} + \lambda\epsilon_{kk}\delta_{ij}$$

$$\sigma_{ii} = 2\mu\epsilon_{ii} + \lambda\epsilon_{kk}\delta_{ii}$$

$$\sigma_{ii} = (3\lambda + 2\mu)\epsilon_{ii}$$

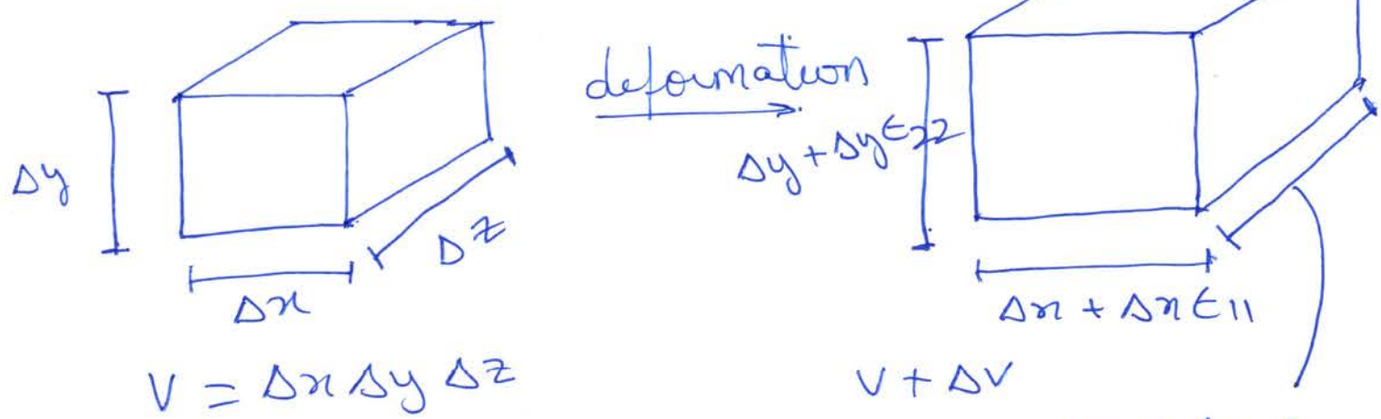
i.e. $-3p = (3\lambda + 2\mu)\epsilon_{ii}$

i.e. $p = -(3\lambda + 2\mu)\frac{1}{3}\epsilon_{ii}$

Define :- $K = \frac{3\lambda + 2\mu}{3}$

i.e.

$$p = -K\epsilon_{ii} \qquad \text{—(i)}$$




$V = \Delta x \Delta y \Delta z$

$V + \Delta V$

$$V + \Delta V = (\Delta x + \Delta x\epsilon_{11})(\Delta y + \Delta y\epsilon_{22})(\Delta z + \Delta z\epsilon_{33})$$

$$= \Delta x \Delta y \Delta z + \Delta x \Delta y \Delta z(\epsilon_{11} + \epsilon_{22} + \epsilon_{33})$$

$$V + \Delta V = V + V\epsilon_{ii} + \text{Higher order terms.}$$

i.e. $\epsilon_{ii} = \frac{\Delta V}{V}$ $\left(\leftrightarrow \epsilon = \frac{\Delta L}{L_0}\right)$.

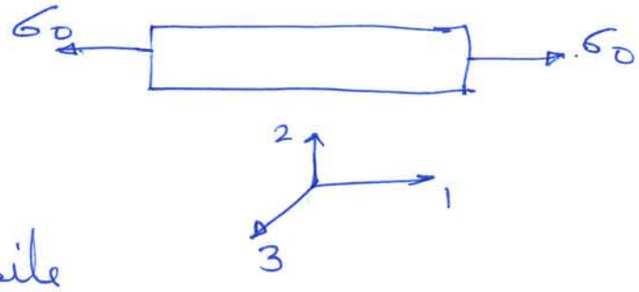
$$P = -K \frac{\Delta V}{V} \quad ; \quad K = \frac{1}{3}(3\lambda + 2\mu) = \frac{1}{3}\frac{\sigma_{ii}}{\epsilon_{ii}}$$

II) Simple Tension

$\sigma_{11} = \sigma_0$

All other $\sigma_{ij} = 0$

$\sigma_0$ is uniaxial tensile stress.




Define :- $\frac{\sigma_{11}}{\epsilon_{11}} = E \rightarrow$ Young's Modulus.

Stress - strain relationship :-

$$\sigma_{ij} = 2\mu\epsilon_{ij} + \lambda\epsilon_{kk}\delta_{ij}$$

$$= 2\mu\epsilon_{ij} + \lambda[\epsilon_{11} + \cancel{\epsilon_{22}}^{0} + \epsilon_{33}]\delta_{ij}$$

$i \neq j \rightarrow \delta_{ij} = 0$

i.e. $\sigma_{ij} = 2\mu\epsilon_{ij}$

i.e. $\sigma_{ij} = 0 \quad \forall\ i \neq j$

Normal strain $\epsilon_{11}$ is

$$2\mu\epsilon_{11} = \sigma_{11} - \frac{\lambda}{3\lambda+2\mu}(\sigma_{11} + \cancel{\sigma_{22}}^{0} + \cancel{\sigma_{33}}^{0})$$

$$2\mu\epsilon_{11} = \frac{2(\lambda+\mu)}{3\lambda+2\mu}\sigma_{11}$$

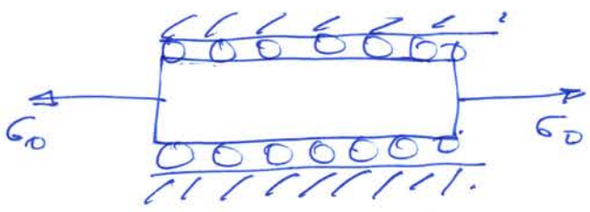
i.e. $$E = \frac{\sigma_{11}}{\epsilon_{11}} = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu}$$

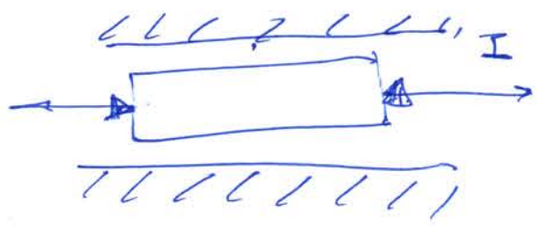
For simple tension : ~~$\sigma_{22} = \sigma_{33} = 0$~~ $\epsilon_{22} \neq \epsilon_{33} \neq 0$.

$$\epsilon_{22} = \epsilon_{33} = \frac{1}{2\mu}\left[0 - \frac{\lambda}{3\lambda+2\mu}\epsilon_{11}\right]$$

$$= -\frac{\lambda}{2(\lambda+\mu)}\epsilon_{11}$$

Poisson's ratio : $$\nu = -\frac{\epsilon_{22}}{\epsilon_{11}} = \frac{\lambda}{2(\lambda+\mu)}$$




$\nu = 0$ , i.e. $\epsilon_{22} = 0$

## IVI. Pure/simple shear

$$\sigma_{ij} = \begin{bmatrix} 0 & \mu k & 0 \\ \mu k & -\mu k^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(simple shear)

small deformation $k << 1$

ignore $k^2$

$= 0$

in x-y plane

$$\sigma_{12} = \sigma_{21} = \tau_0$$

$$\sigma_{ij} = 0 \quad \text{— all other } \sigma_{ij}$$

Then $\sigma_{12} = 2\mu \epsilon_{12}$

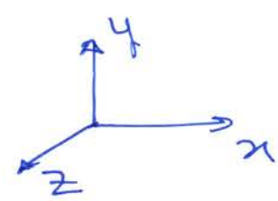
i.e. $\mu = \dfrac{\sigma_{12}}{2\epsilon_{12}} = G$ shear modulus.

$$= \frac{\sigma_{12}}{\gamma_{12}}$$

$2\epsilon_{12} = \gamma_{12} \rightarrow$ Engineering shear strain.

Summary:

Expanded form of Hooke's law:




① strain in terms of stress

$$\epsilon_{xx} = \frac{1}{E}\left[\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})\right]$$

$$\epsilon_{yy} = \frac{1}{E}\left[\sigma_{yy} - \nu(\sigma_{zz} + \sigma_{xx})\right]$$

$$\epsilon_{zz} = \frac{1}{E}\left[\sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy})\right]$$

$$2\epsilon_{xy} = \frac{2(1+\nu)}{E}\sigma_{xy}$$

$$2\epsilon_{yz} = \frac{2(1+\nu)}{E}\sigma_{yz}$$

$$2\epsilon_{xz} = \frac{2(1+\nu)}{E}\sigma_{xz}$$

(2) stress in terms of strains.

$$\sigma_{xx} = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\epsilon_{xx} + \nu(\epsilon_{yy} + \epsilon_{zz})\right]$$

$$\sigma_{yy} = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\epsilon_{yy} + \nu(\epsilon_{xx} + \epsilon_{zz})\right]$$

$$\sigma_{zz} = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\epsilon_{zz} + \nu(\epsilon_{xx} + \epsilon_{yy})\right]$$

$$\sigma_{xy} = \frac{E}{2(1+\nu)}2\epsilon_{xy}, \quad \sigma_{yz} = \frac{E}{2(1+\nu)}2\epsilon_{yz}$$

$$\sigma_{xz} = \frac{E}{2(1+\nu)}2\epsilon_{xz}.$$

- Poisson's ratio:

$$\bar{\nu} = \frac{E}{(1+\nu)(1-2\nu)}\left[\epsilon_1^2 + \epsilon_2^2 + \epsilon_3^2 + 2\nu(\epsilon_1\epsilon_2 + \epsilon_2\epsilon_3 + \epsilon_3\epsilon_1)\right]$$

If $\bar{\nu}$ is to remain +ve

$$(1+\nu)(1-2\nu) > 0$$

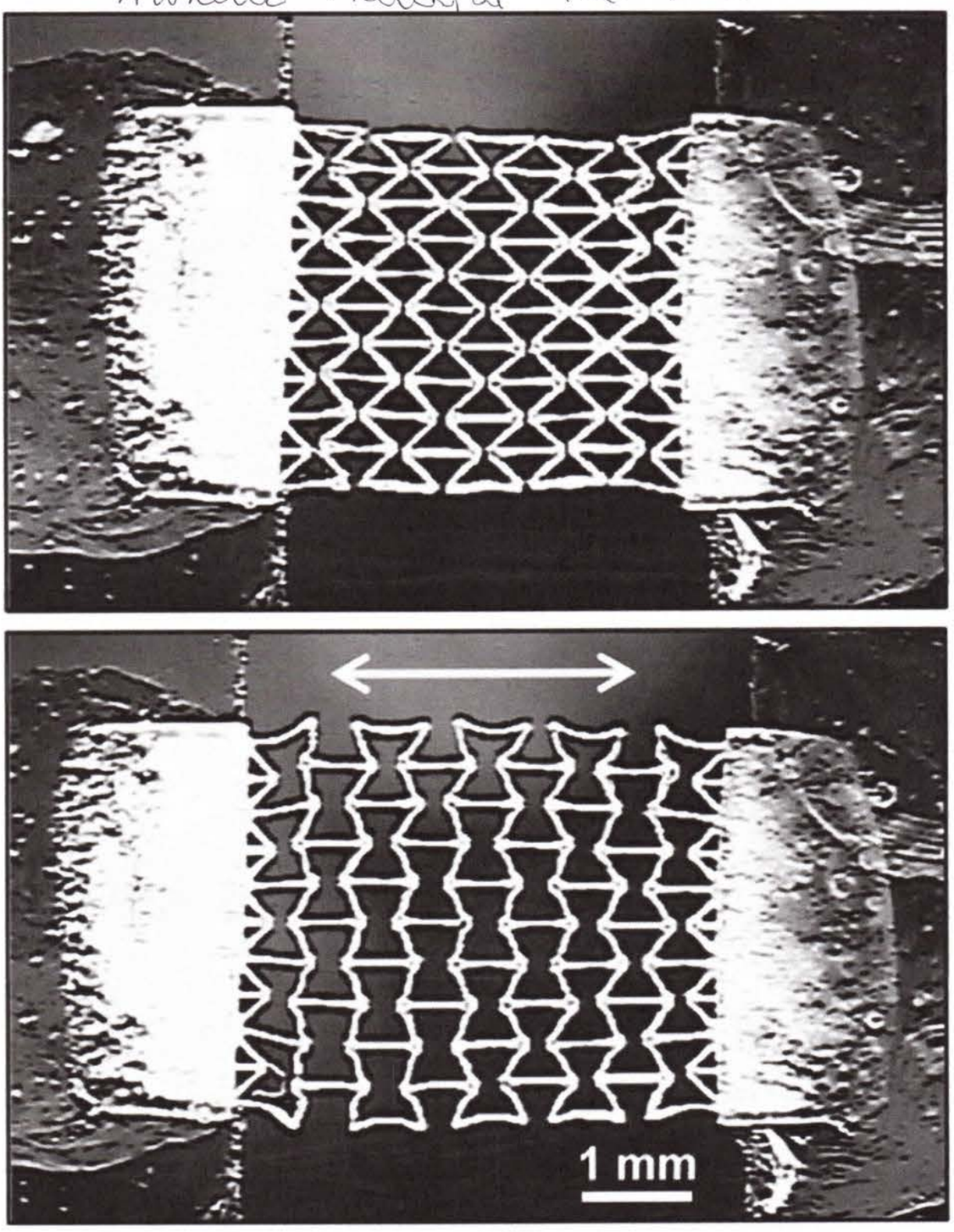
i.e. $\nu > -1$, or $\nu < 0.5$.

$$\nu = (-1, 0.5)$$

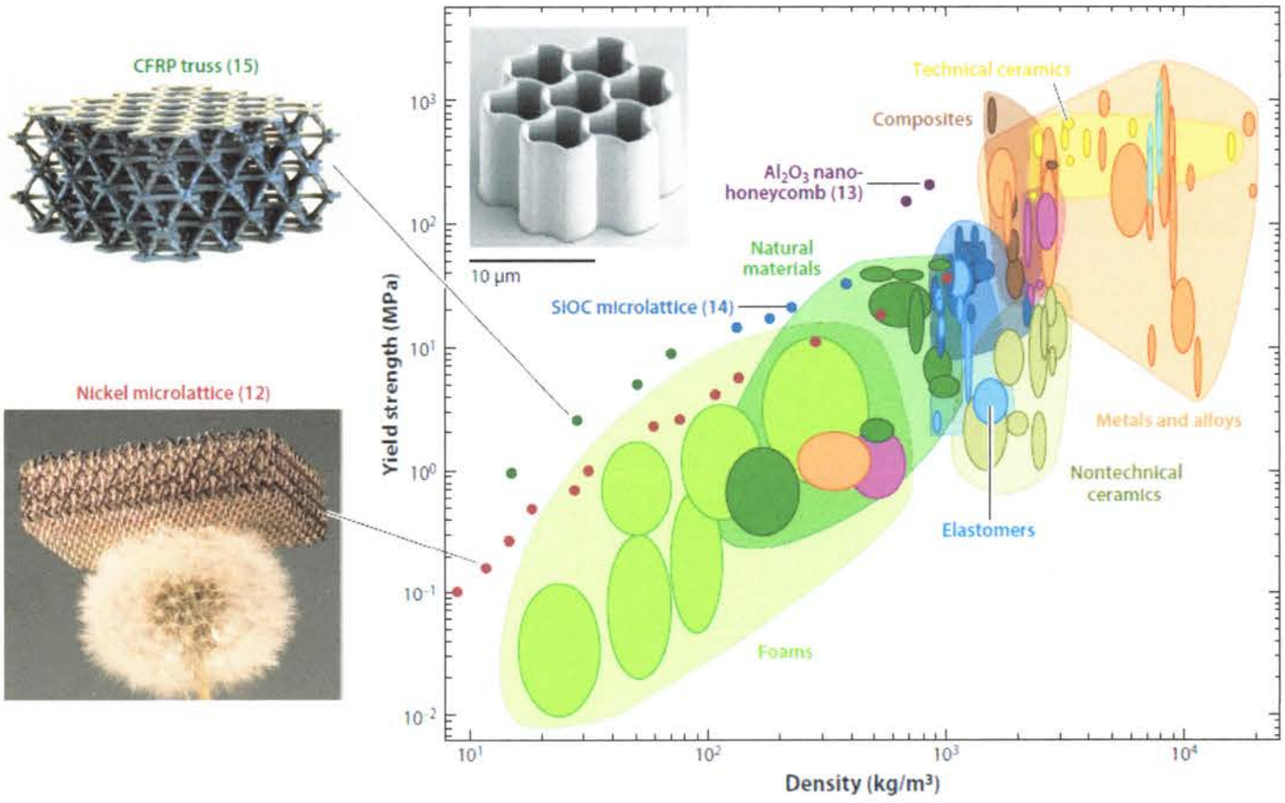
$\nu = 0.5$ typically called incompressible material.

$J = 1 \rightarrow$ True condition of incompressibility

Auxetic Material i.e. $\nu < 0$



Architected materials



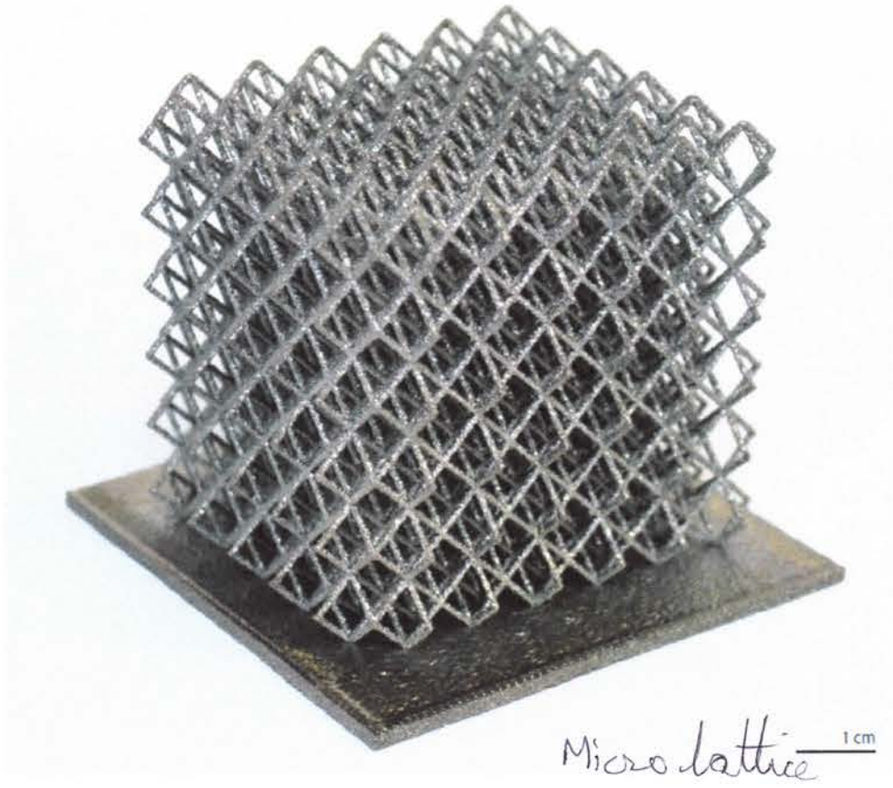
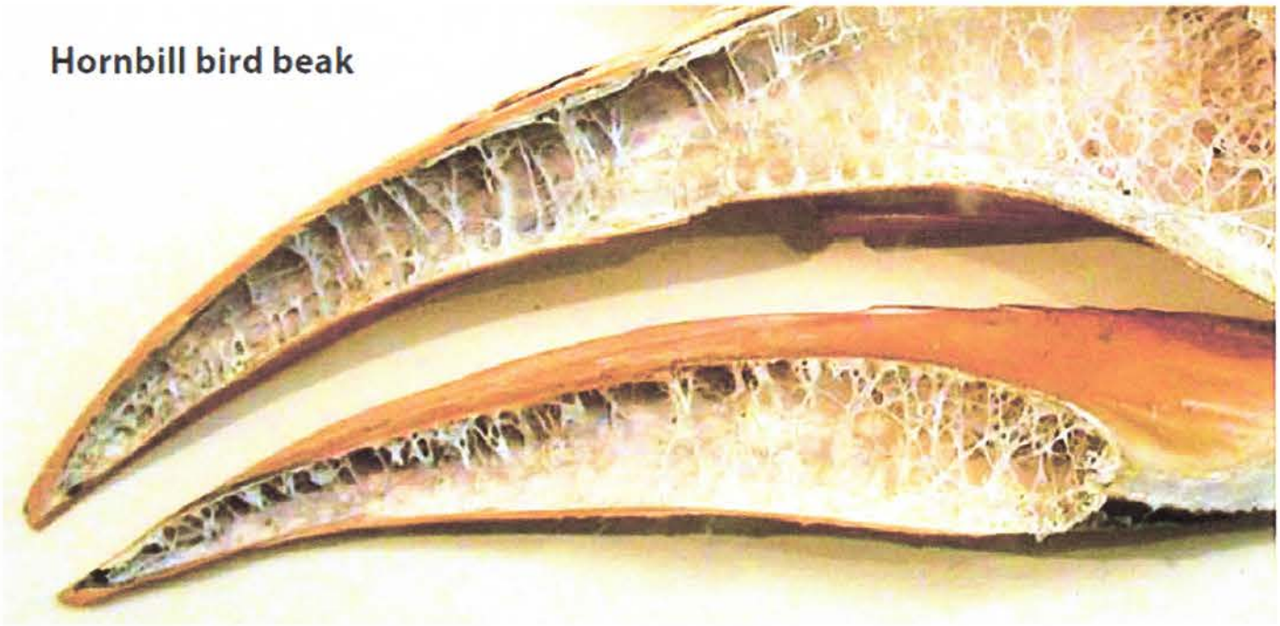




Figure 11

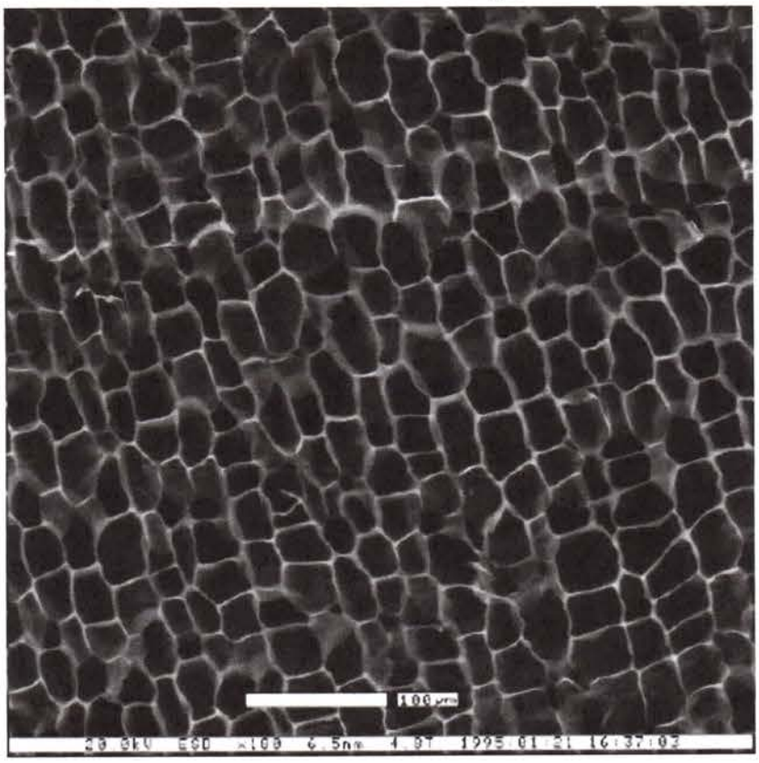
Ti-Al6-V4 lattice material fabricated via electron beam melting (by CalRAM, Inc., for HRL, LLC).

**Hornbill bird beak**



**Bird wing bone**

cork microstructure



Hierarchical materials

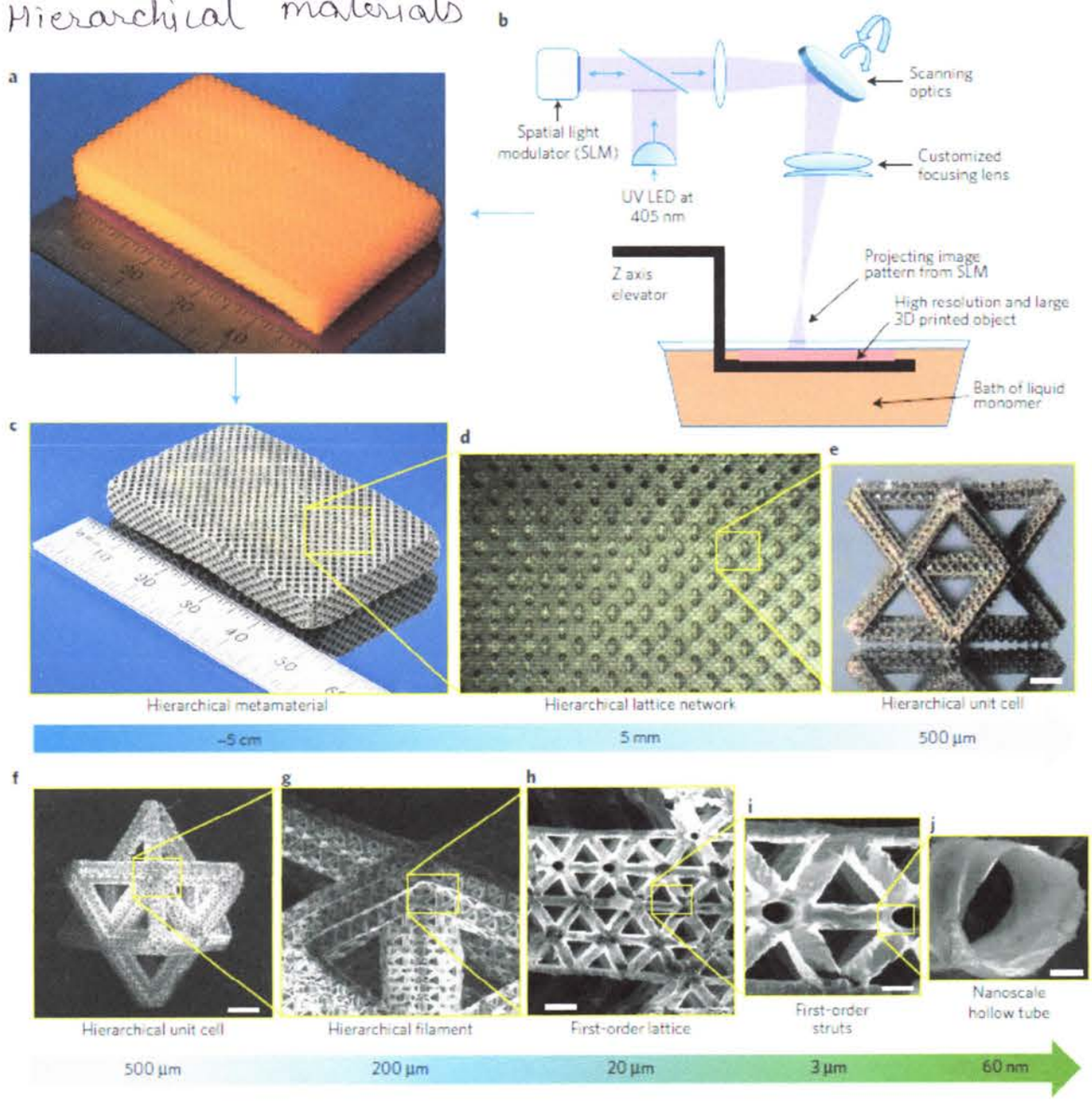



**Figure 1 | Nickel alloy hierarchical metamaterial and critical features across seven orders of magnitude in length scale. a**, Polymer metamaterial template. **b**, Large-area, high-resolution additive manufacturing of hierarchical metamaterials. **c–e**, Optical microscope images of bulk hierarchical lattice material with a network of hierarchical stretch-dominated octet unit cells. Scale bar in **e**: 80 µm. **f–j**, Scanning electron micrographs showing cross-section breakdown of structural hierarchy of the multiscale metamaterial unit cell shown in **e** down to tens of nanometres in wall thickness. Scale bar in **h**: 10 µm; scale bar in **i**: 3 µm.

# Chapter 5 : Linear Elasticity Problems.

## 5.1 Reduction to linear theory of elasticity

### ✱ General Elasticity formulation:

I] Equilibrium :- $\frac{\partial}{\partial x_K^0}\left[\frac{\partial x_i}{\partial x_j^0}\sigma_{Kj}^K\right] + \rho^0 b_i^0 = \rho^0 \ddot{u}_i$ — 3 Eq's

$$\sigma_{Kj}^K = \sigma_{jK}^K$$

II] Strain - displacement :

$$2E_{ij} = \frac{\partial u_i}{\partial x_j^0} + \frac{\partial u_j}{\partial x_i^0} + \frac{\partial u_K}{\partial x_i^0}\frac{\partial u_K}{\partial x_j^0}$$ — 6 Eq's.

III] Constitutive relationship :- — 6 Eq's.

$$\sigma_{ij}^K = \frac{\partial U}{\partial E_{ij}}$$

Field.
15 Equations, 15 unknowns (3 displacements, 6 strains, 6 stresses)

↓

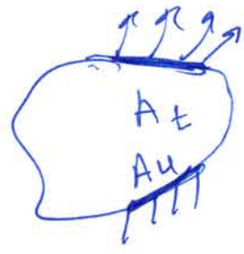
Differential equations.

↳ integration leads to integration const.

IV] Boundary conditions:

specify : $t_i^0 = \sigma_{ji}^K n_j^0$ on $A_t$

$u_i$ on $A_u$



(I - IV) are applicable to general elasticity problems, i.e. large deformation as well as non-linear material behavior.

★ Linear Theory of Elasticity (Classical)

- small deformations
- Material behavior is linear
- Ignore inertial effects (i.e. $\ddot{u}_i = 0$)

The equations can be rewritten as:-

$$\frac{\partial \sigma_{ij}}{\partial x_j} + \rho b_i = 0 \quad , \quad \sigma_{ij} = \sigma_{ji} \qquad \text{— (1)}$$

$$\epsilon_{ij} = \frac{1}{2}\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right] \qquad \text{— (2)}$$

$$\sigma_{ij} = 2\mu \epsilon_{ij} + \lambda \epsilon_{kk} \delta_{ij} \qquad \text{— (3).}$$

specify : $u_i$ or $t_i$ for boundary conditions.

still 15 eq, & 15 unknowns.

Since $\sigma_{ij}$ & $\epsilon_{ij}$ are related and $\epsilon_{ij}$ & $u_i$ are related, we can reduce the number of independent unknowns.

We can take two routes :-

(1) Displacement formulation (Navier).

Take displacements as the basic unknowns.

i.e. write everything in terms of $u_i$.

Plug (2) into (3) & (3) into (1).

we get :-

$$\mu \frac{\partial^2 u_i}{\partial x_k \partial x_k} + (\lambda+\mu)\frac{\partial^2 u_k}{\partial x_i \partial x_k} + \rho b_i = 0$$ — 3 Equations, 3 unknowns

Boundary conditions :-

on $A_u$ = $u_i$

on $A_t$ : $\lambda \frac{\partial u_k}{\partial x_k} n_i + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) n_j = t_i$

* Navier's Equations.

(2) Stress Formulation (Beltrami - Mitchell)

Assume $\sigma_{ij}$ as the basic unknowns.

at each point, 6 unknowns. $\{\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{23}, \sigma_{13}, \sigma_{12}\}$

— Must satisfy equilibrium. ← 3 equation only!

Assume $\sigma_{ij}$ —Constitutive Eq→ $\epsilon_{ij}$ —integrate→ $u_i$ (3 unkn

$$2\epsilon_{ij} = \frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}$$ (A) → 6 equation

Differentiate (A) twice.

$$2\frac{\partial^2 \epsilon_{ij}}{\partial x_k \partial x_l} = \frac{\partial^3 u_i}{\partial x_j \partial x_k \partial x_l} + \frac{\partial^3 u_j}{\partial x_i \partial x_k \partial x_l}$$

i.e. $$\frac{\partial^2 \epsilon_{ij}}{\partial x_k \partial x_l} + \frac{\partial^2 \epsilon_{kl}}{\partial x_i \partial x_j} - \frac{\partial^2 \epsilon_{ik}}{\partial x_j \partial x_l} - \frac{\partial^2 \epsilon_{jl}}{\partial x_i \partial x_k} = 0$$

81 equations relating $\epsilon_{ij}$'s

These constraint possible $\epsilon_{ij}$ values.

Only 6 equations are non-trivial & independent

$$\frac{\partial^2 \epsilon_{xx}}{\partial y \partial z} = \frac{\partial}{\partial x}\left[-\frac{\partial \epsilon_{yz}}{\partial x} + \frac{\partial \epsilon_{zx}}{\partial y} + \frac{\partial \epsilon_{xy}}{\partial z}\right]$$

$$\frac{\partial^2 \epsilon_{yy}}{\partial x \partial z} = \frac{\partial}{\partial y}\left[-\frac{\partial \epsilon_{xz}}{\partial y} + \frac{\partial \epsilon_{xy}}{\partial z} + \frac{\partial \epsilon_{yz}}{\partial x}\right]$$

$$\frac{\partial^2 \epsilon_{zz}}{\partial x \partial y} = \frac{\partial}{\partial z}\left[-\frac{\partial \epsilon_{xy}}{\partial z} + \frac{\partial \epsilon_{yz}}{\partial x} + \frac{\partial \epsilon_{zx}}{\partial y}\right]$$

$$2\frac{\partial^2 \epsilon_{xy}}{\partial x \partial y} = \frac{\partial^2 \epsilon_{xx}}{\partial y^2} + \frac{\partial^2 \epsilon_{yy}}{\partial x^2} \quad \leftarrow \star.$$

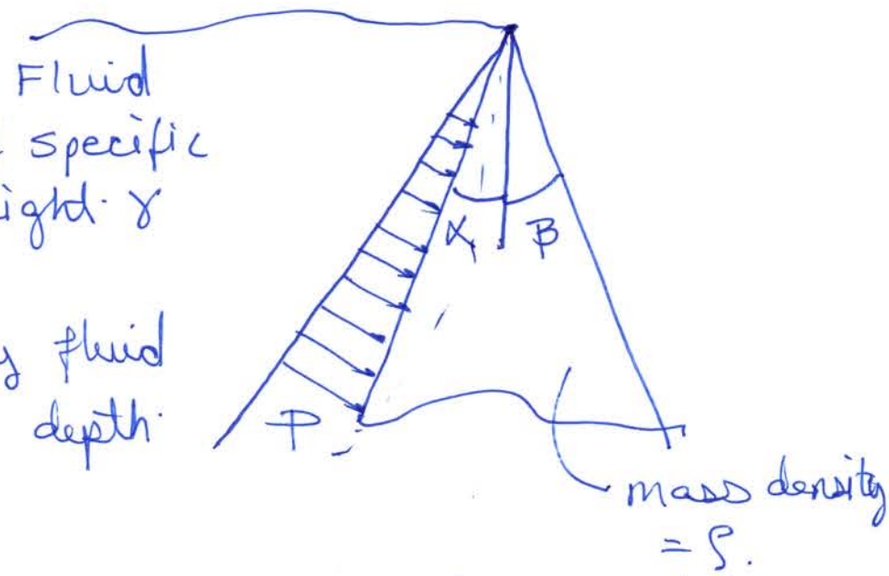
$$2\frac{\partial^2 \epsilon_{yz}}{\partial y \partial z} = \frac{\partial^2 \epsilon_{yy}}{\partial z^2} + \frac{\partial^2 \epsilon_{zz}}{\partial x^2}$$

$$2\frac{\partial^2 \epsilon_{zx}}{\partial z \partial x} = \frac{\partial^2 \epsilon_{zz}}{\partial x^2} + \frac{\partial^2 \epsilon_{xx}}{\partial z^2}.$$

— 6 equations → Equations of compatibility.

Example :- Lewis Problem for Masonry dams.

2-D.




* We know that pressure exerted by fluid varies linearly with depth. ($P \propto y$)

Therefore, we can assume the stress field to be linear.

I] stress field :-

Assume :-

$$\sigma_{xx} = a_1 x + b_1 y + c_1$$

$$\sigma_{yy} = a_2 x + b_2 y + c_2$$

$$\sigma_{xy} = a_{12} x + b_{12} y + c_{12}$$

9 unknown

- Needs to satisfy equilibrium
- Compatibility equations.
- Boundary conditions.

- Let origin be at apex.

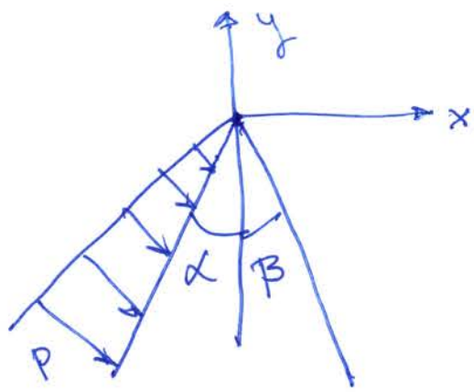
@ $x=0$ & $y=0$.

$$0 = a_1(0) + b_1(0) + c_1$$

$$0 = a_2(0) + b_2(0) + c_2$$

$$0 = a_{12}(0) + b_{12}(0) + c_{12}$$

i.e. $c_1 = c_2 = c_{12} = 0$.



- Equilibrium :-

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \rho b_x = 0 \quad \text{i.e. } a_1 + b_{12} + 0 = 0$$

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \rho b_y = 0 \quad \text{i.e. } a_{12} + b_2 - \rho g = 0$$

↓ due to self-weight

i.e. $b_{12} = -a_1$

$a_{12} = \rho g - b_2$

i.e. $\sigma_{xx} = a_1 x + b_1 y$

$\sigma_{yy} = a_2 x + b_2 y$

$\sigma_{xy} = (\rho g - b_2) x - a_1 y = -b_2 x - a_1 y + \rho g x$

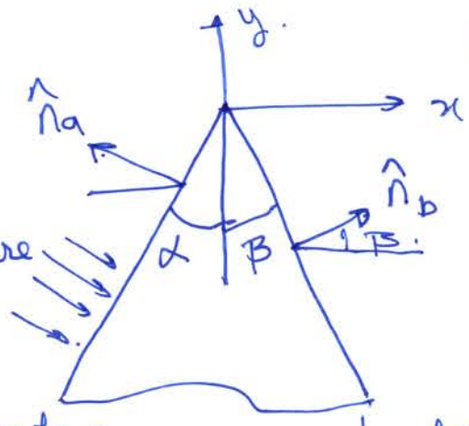
$a_1, a_2, b_1, b_2 \longrightarrow$ 4 independent unknowns.

- Compatibility :-

$$\frac{\partial^2 \epsilon_{xy}}{\partial x \partial y} = \frac{\partial^2 \epsilon_{xx}}{\partial y^2} + \frac{\partial^2 \epsilon_{yy}}{\partial x^2}$$

$\therefore$ $\sigma$'s are linear, compatibility is automatically satisfied.

II] Boundary conditions:




In general :-

$$t_x = \sigma_{xx} n_x + \sigma_{yx} n_y + \sigma_{zx} n_z$$

$$t_y = \sigma_{xy} n_x + \sigma_{yy} n_y + \sigma_{zy} n_z$$

For b-face :-

$$\{n_x, n_y, n_z\} = \{\cos\beta, \sin\beta, 0\}$$

i.e. $t_x = \sigma_{xx}\cos\beta + \sigma_{yx}\sin\beta + 0 = 0$

$t_y = \sigma_{xy}\cos\beta + \sigma_{yy}\sin\beta + 0 = 0$

i.e. $t_x = (a_1 x + b_1 y)\cos\beta + (-b_2 x - a_1 y + \rho g x)\sin\beta = 0$

$t_y = (-b_2 x - a_1 y + \rho g x)\cos\beta + (a_2 x + b_2 y)\sin\beta = 0$

On b-face :- $\tan\beta = \dfrac{x}{-y}$ i.e. $x = -y\tan\beta$

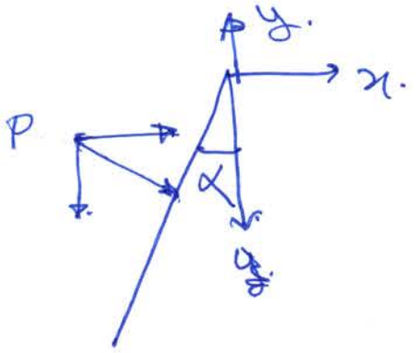
Plug in to get:

$$b_1 + b_2\tan^2\beta - 2a_1\tan\beta = \rho g \tan^2\beta \quad (1)$$

$$a_1 + a_2\tan^2\beta + 2b_2\tan\beta = -\rho g \tan\beta \quad (2)$$

P.T.O

For a- face-




$\{n_x, n_y, n_z\} = \{-\cos\alpha, \sin\alpha, 0\}$

$x = y\tan\alpha,$

$P = -\gamma y$

$t_x = p\cos\alpha = -\gamma y\cos\alpha$

$t_y = -p\sin\alpha = +\gamma y\sin\alpha$

$t_x = \sigma_{xx}(-\cos\alpha) + \sigma_{xy}(\sin\alpha) = -\gamma y\cos\alpha$

$t_y = \sigma_{xy}(-\cos\alpha) + \sigma_{yy}(\sin\alpha) = \gamma y\sin\alpha$

(3) i.e. $b_1 + b_2\tan^2\alpha + 2a_1\tan\alpha = -\gamma + \rho g\tan^2\alpha$

(4) $a_1 + a_2\tan^2\alpha + 2b_2\tan\alpha = -\gamma\tan\alpha + \rho g\tan\alpha$

Solve to get $a_1, a_2, b_1$ & $b_2$.

Last class:

- linear theory of elasticity:

Equilibrium : $\frac{\partial \sigma_{ij}}{\partial x_j} + \rho b_i = 0$ , $\sigma_{ij} = \sigma_{ji}$

strain displacement : $\epsilon_{ij} = \frac{1}{2}\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right]$

Constitutive eq : $\sigma_{ij} = 2\mu \epsilon_{ij} + \lambda \epsilon_{kk} \delta_{ij}$

Boundary Cond : $u_i$ on $A_u$

$t_i$ on $A_t$

- Displacement formulation: Everything in terms of $u_1, u_2$ & $u_3$.

Navier's equation:

$$\mu \frac{\partial^2 u_i}{\partial x_k \partial x_k} + (\lambda + \mu) \frac{\partial^2 u_k}{\partial x_i \partial x_k} + \rho b_i = 0$$

B.C. : on $A_u$ : $u_i$

on $A_t$ : $\lambda \frac{\partial u_k}{\partial x_k} n_i + \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) n_j = t_i$

- stress formulation: Everything in terms of $\sigma_{ij}$.

- Satisfy <u>equilibrium</u>
- single valued disp → <u>Compatibility</u> equations

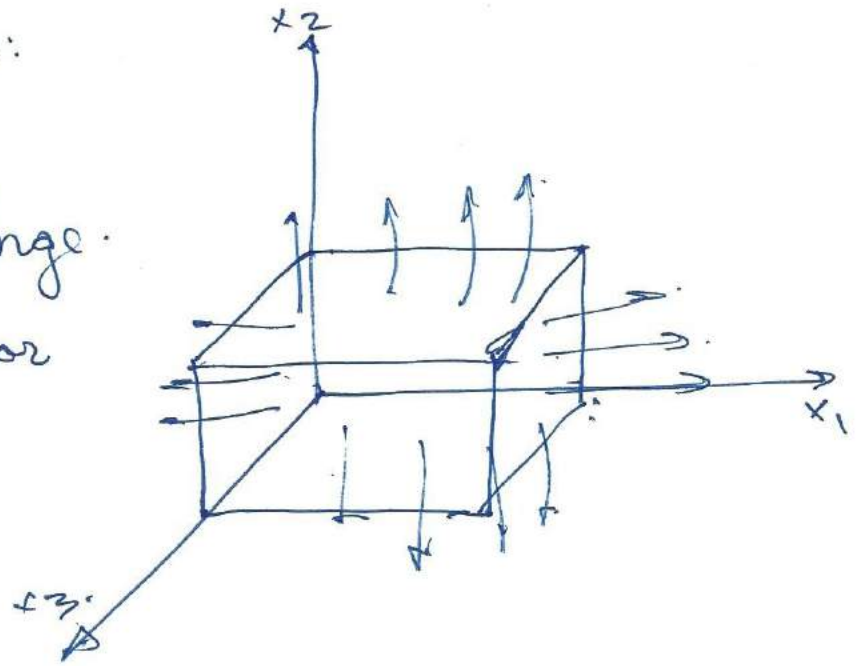
$$\frac{\partial^2 \sigma_{ij}}{\partial x_k \partial x_l} + \frac{\partial^2 \epsilon_{kl}}{\partial x_i \partial x_j} - \frac{\partial^2 \epsilon_{ik}}{\partial x_j \partial x_l} - \frac{\partial^2 \epsilon_{jl}}{\partial x_i \partial x_k} = 0$$

only 6 are independent / + Boundary conditions.

## 5.2 Plane problems (2-D).

Two formulations:

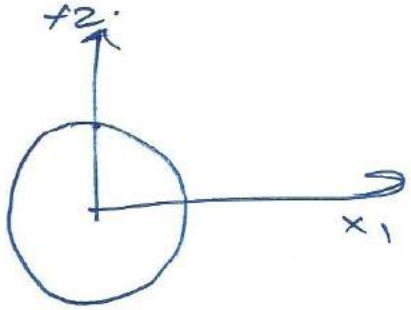
- Thickness along $x_3$ does not change.
- Loads along $x_1$ or $x_2$ direction.

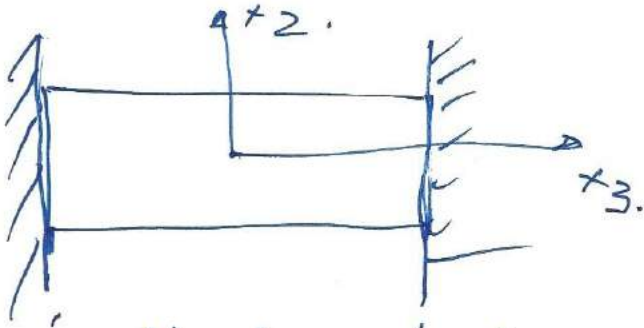



(i) Plane strain $\parallel^{el}$ $x_1-x_2$ plane.

Assume: $u_3 = 0$, i.e. $\epsilon_{33} = \epsilon_{23} = \epsilon_{13} = 0$

$\epsilon_{12} \neq \epsilon_{11} \neq \epsilon_{22} \neq 0$.




smooth boundaries.

3-D: $\sigma_{ij} = 2\mu\epsilon_{ij} + \lambda\epsilon_{kk}\delta_{ij}$

$i, j, k = 1, 2, 3$.

Plain strain:-

$$\sigma_{11} = 2\mu\epsilon_{11} + \lambda[\epsilon_{11} + \epsilon_{22} + \cancel{\epsilon_{33}}^{\,0}]$$

$$\sigma_{22} = 2\mu\epsilon_{22} + \lambda[\epsilon_{11} + \epsilon_{22} + \cancel{\epsilon_{33}}^{\,0}]$$

$$\sigma_{33} = 2\mu\epsilon_{33}^{\,0} + \lambda\left[\epsilon_{11} + \epsilon_{22} + \epsilon_{33}^{\,0}\right]$$

$$\sigma_{33} = \lambda\left[\epsilon_{11} + \epsilon_{22}\right] \neq 0$$

$$\sigma_{12} = 2\mu\epsilon_{12}$$

$$\sigma_{13} = 2\mu\epsilon_{13}^{\,0} = 0$$

$$\sigma_{23} = 2\mu\epsilon_{23}^{\,0} = 0$$

We can write :-

$$\sigma_{\alpha\beta} = 2\mu\epsilon_{\alpha\beta} + \lambda(\epsilon_{nn}\delta_{\alpha\beta}), \quad \alpha,\beta,n = 1,2.$$

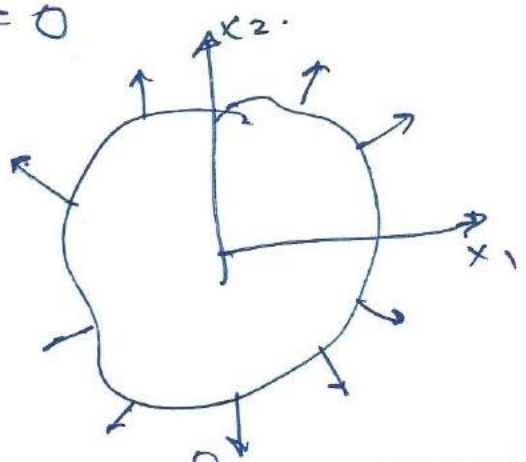
Invert to get :-

$$\epsilon_{\alpha\beta} = \frac{1}{2\mu}\left[\sigma_{\alpha\beta} - \frac{3-K}{4}\sigma_{nn}\delta_{\alpha\beta}\right], \quad \alpha,\beta,n = 1,2.$$

$$K = 3 - 4\nu$$

(ii) Plane stress

Assume :- $\sigma_{13} = \sigma_{23} = \sigma_{33} = 0$




Stress-strain relations :

$$\epsilon_{\alpha\beta} = \frac{1}{2\mu}\left[\sigma_{\alpha\beta} - \frac{\nu}{1+\nu}\sigma_{nn}\delta_{\alpha\beta}\right]$$

$$\alpha,\beta,n = 1,2.$$

$$\epsilon_{13} = \epsilon_{23} = 0.$$

$$\epsilon_{33} = \frac{1}{2\mu}\left[\sigma_{33}^{\,0} - \frac{\nu}{1+\nu}(\sigma_{11} + \sigma_{22} + \sigma_{33}^{\,0})\right]$$

$\nu \rightarrow$ Poisson ratio

i.e. $$\epsilon_{33} = \frac{-\nu}{2\mu(1+\nu)}(\sigma_{11} + \sigma_{22})$$

Compatibility in 2-D.

$$\frac{\partial^2 \epsilon_{11}}{\partial x_2^2} + \frac{\partial^2 \epsilon_{22}}{\partial x_1^2} = 2\frac{\partial^2 \epsilon_{12}}{\partial x_1 \partial x_2}$$

Since we cannot satisfy compatibility exactly, plane stress solutions are approximate.

---

- Compatibility in terms of stresses.

$$\frac{\partial^2 \epsilon_{11}}{\partial x_2^2} + \frac{\partial^2 \epsilon_{22}}{\partial x_1^2} = 2\frac{\partial^2 \epsilon_{12}}{\partial x_1 \partial x_2} \quad (1)$$

We have:

$$\epsilon_{11} = \frac{1}{2\mu}\left[\sigma_{11} - \frac{3-\kappa}{4}(\sigma_{11} + \sigma_{22})\right]$$

$$\epsilon_{22} = \frac{1}{2\mu}\left[\sigma_{22} - \frac{3-\kappa}{4}(\sigma_{11} + \sigma_{22})\right] \quad (2)$$

$$\epsilon_{12} = \frac{1}{2\mu}\sigma_{12}$$

$$\kappa = \begin{cases} 3 - 4\nu & \text{plane strain} \\ \dfrac{3-\nu}{1+\nu} & \text{plane stress} \end{cases}$$

Plug (2) into (1)

$$\frac{\partial^2 \sigma_{11}}{\partial x_2^2} - \frac{3-\kappa}{4}\frac{\partial^2}{\partial x_2^2}(\sigma_{11}+\sigma_{22}) + \frac{\partial^2 \sigma_{22}}{\partial x_1^2}$$

$$- \frac{3-\kappa}{4}\frac{\partial^2}{\partial x_1^2}(\sigma_{11}+\sigma_{22}) - \frac{2\partial^2 \sigma_{12}}{\partial x_1 \partial x_2} = 0 \qquad (3)$$

From equilibrium :-

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + b_1 = 0 \qquad (4)$$

$$\frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + b_2 = 0 \qquad (5)$$

$$\frac{\partial}{\partial x_1}(4) : \frac{\partial^2 \sigma_{12}}{\partial x_1 \partial x_2} = -\frac{\partial^2 \sigma_{11}}{\partial x_1^2} - \frac{\partial b_1}{\partial x_1} \qquad (4')$$

$$\frac{\partial}{\partial x_2}(5) : \frac{\partial^2 \sigma_{12}}{\partial x_1 \partial x_2} = -\frac{\partial^2 \sigma_{22}}{\partial x_2^2} - \frac{\partial b_2}{\partial x_2} \qquad (5')$$

Add 4' & 5'

$$2\frac{\partial^2 \sigma_{12}}{\partial x_1 \partial x_2} = -\left[\frac{\partial^2 \sigma_{11}}{\partial x_1^2} + \frac{\partial^2 \sigma_{22}}{\partial x_2^2} + \frac{\partial b_1}{\partial x_1} + \frac{\partial b_2}{\partial x_2}\right] \qquad (6)$$

Use (6) in (3).

$$\nabla^2 \sigma_{11} + \nabla^2 \sigma_{22} - \frac{3-\kappa}{4}\nabla^2(\sigma_{11}+\sigma_{22}) + \frac{\partial b_1}{\partial x_1} + \frac{\partial b_2}{\partial x_2} = 0$$

$$\nabla^2 = \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right]$$

Rearrange to get:

$$\frac{1+\kappa}{4}\nabla^2(\sigma_{11}+\sigma_{22}) + \frac{\partial b_1}{\partial x_1} + \frac{\partial b_2}{\partial x_2} = 0$$

i.e. $$\nabla^2(\sigma_{11}+\sigma_{22}) = \frac{-4}{1+\kappa}\left(\frac{\partial b_1}{\partial x_1} + \frac{\partial b_2}{\partial x_2}\right) \quad \star.$$

If body forces are absent

$$\nabla^2(\sigma_{11}+\sigma_{22}) = 0$$

If body force are conservative; then

$$b_1 = -\frac{\partial V}{\partial x_1}, \quad b_2 = -\frac{\partial V}{\partial x_2}, \quad V \equiv V(x,y)$$

↓
Potential.

then $\star$ becomes:-

$$\boxed{\nabla^2(\sigma_{11}+\sigma_{22}) = \frac{4}{\kappa+1}\nabla^2 V}$$

Compatibility Equation.

+ Governing equations for plane problems:

- Equilibrium equation:

$$\sigma_{\alpha\beta,\beta} + b_\alpha = 0 \qquad ,_\beta = \frac{\partial}{\partial x_\beta}$$

with $$\nabla^2(\sigma_{11}+\sigma_{22}) = \frac{4}{\kappa+1}\nabla^2 V$$

+ specify B.C.

## 5.3 Airy stress function method.

Assume $b_i = 0$

Then compatibility equation becomes:-

$$\nabla^2(\sigma_{11} + \sigma_{22}) = 0 \quad (\star)$$

Assume a "stress function" $\phi(x_1, x_2)$ such that:-

$$\sigma_{11} = \frac{\partial^2 \phi}{\partial x_2^2}, \quad \sigma_{22} = \frac{\partial^2 \phi}{\partial x_1^2}, \quad \sigma_{12} = \frac{-\partial^2 \phi}{\partial x_1 \partial x_2} \quad (\star\star)$$

Then, equilibrium equation becomes

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} = \frac{\partial}{\partial x_1}\left(\frac{\partial^2 \phi}{\partial x_2^2}\right) + \frac{\partial}{\partial x_2}\left(\frac{-\partial^2 \phi}{\partial x_1 \partial x_2}\right)$$

$$= \frac{\partial^3 \phi}{\partial x_1 \partial x_2^2} - \frac{\partial^3 \phi}{\partial x_1 \partial x_2^2}$$

$$= 0 \; !$$

similarly: $\frac{\partial \sigma_{21}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} = 0 \; !$

Thus, $\phi$ satisfies equilibrium automatically.

Substitute $(\star\star)$ into $(\star)$.

i.e. $$\nabla^2(\sigma_{11} + \sigma_{22}) = 0$$

$$\nabla^2\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right)\phi = 0$$

i.e. $$\nabla^2 \nabla^2 \phi = 0$$

$$\nabla^4 \phi = 0$$ biharmonic equation

i.e. $\phi$ is biharmonic.

$$\nabla^2\nabla^2\phi = \left(\frac{\partial^4}{\partial x_1^4} + 2\frac{\partial^4}{\partial^2 x_1^2 \partial^2 x_2^2} + \frac{\partial^4}{\partial x_2^4}\right)\phi$$

If body forces are present:

$$b_1 = -\frac{\partial V}{\partial x_1} \quad \& \quad b_2 = -\frac{\partial V}{\partial x_2}$$, then $V(x_1, x_2)$

$$\sigma_{11} = \frac{\partial^2 \phi}{\partial x_2^2} + V$$

$$\sigma_{22} = \frac{\partial^2 \phi}{\partial x_1^2} + V$$

$$\sigma_{12} = -\frac{\partial^2 \phi}{\partial x_1 \partial x_2}$$

Compatibility becomes:

$$\nabla^2\nabla^2\phi = -\frac{2(K-1)}{1+K}\nabla^2 V$$ inhomogeneous

General solution: $\phi = \phi_c + \phi_p$

where: $\nabla^2\nabla^2\phi_c = 0$, $\nabla^2\nabla^2\phi_p = -\frac{2(K-1)}{1+K}\nabla^2 V$

i.e. $\phi_c$ is biharmonic while $\phi_p$ depends on body forces.

$\phi \rightarrow$ Airy stress function.

Last class:-

- For plane problems:
  - Equilibrium equations:

$$\sigma_{\alpha\beta,\beta} + b_\alpha = 0$$

with $$\nabla^2(\sigma_{11}+\sigma_{22}) = \frac{4}{\kappa+1}\nabla^2 V$$

+ Boundary conditions to be satisfied

- Airy stress function:- $\phi(x_1, x_2)$

If no body forces, then

$$\nabla^4\phi = 0 \quad , \text{ and}$$

$$\sigma_{11} = \frac{\partial^2\phi}{\partial x_2^2} \quad , \quad \sigma_{22} = \frac{\partial^2\phi}{\partial x_1^2} \quad , \quad \sigma_{12} = -\frac{\partial^2\phi}{\partial x_1 \partial x_2}$$

If body forces exist & are conservative, i.e.

$$b_1 = -\frac{\partial V}{\partial x_1} \quad \& \quad b_2 = -\frac{\partial V}{\partial x_2} \quad , \text{ then}$$

$$\nabla^4\phi = -\frac{2(\kappa+1)}{1+\kappa}\nabla^2 V \quad , \text{ and}$$

$$\sigma_{11} = \frac{\partial^2\phi}{\partial x_2^2} + V \quad , \quad \sigma_{22} = \frac{\partial^2\phi}{\partial x_1^2} + V$$

$$\& \quad \sigma_{12} = -\frac{\partial^2\phi}{\partial x \partial y}$$

Examples: Satisfy: $\nabla^4 \phi = 0$ 2D :- $(x, y) \equiv (x_1, x_2)$

$$\sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2}, \quad \sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2}, \quad \sigma_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y}.$$

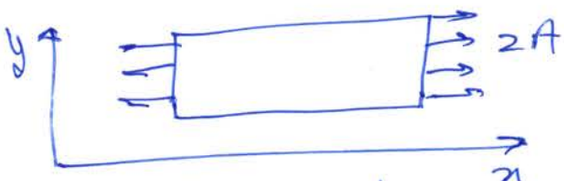
$\nabla^4 \phi = 0$ satisfied by all polynomials of degree 3 or lower.

(i) $\phi = A y^2$, A is a constant.

$\nabla^4 \phi = 0$ ✓

$$\sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2} = 2A$$

$$\sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2} = 0, \quad \sigma_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} = 0$$



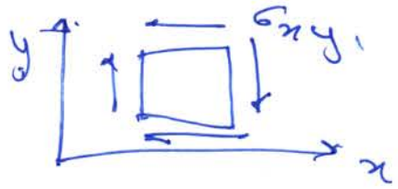

Uniaxial tension.

(ii) $\phi = Bxy$, B is a constant

$\nabla^4 \phi = 0$ ✓

$$\sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2} = 0, \quad \sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2} = 0$$

$$\sigma_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} = -B$$



(iii) $\phi = Ay^2 + Bxy$

$\nabla^4 \phi = 0$ ✓

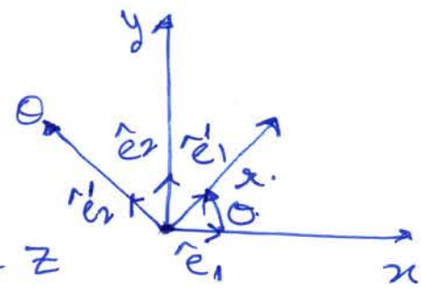
$\sigma_{xx} = 2A$ , $\sigma_{yy} = 0$ , $\sigma_{xy} = -B$.

uniaxial tension + pure shear.

- Cylindrical coordinates :-

$(x, y, z)$ : $u_x, u_y, u_z$

$(r, \theta, z)$ : $u_r, u_\theta, u_z$




$x = r\cos\theta$ , $y = r\sin\theta$ , $z = z$

$r^2 = x^2 + y^2$ , $\tan\theta = \frac{y}{x}$ i.e. $\theta = \tan^{-1}\left(\frac{y}{x}\right)$ , $z = z$

- Derivatives :-

$\frac{\partial r}{\partial x}$ : $x^2 + y^2 = r^2 \Rightarrow 2x = 2r\frac{\partial r}{\partial x}$

$\therefore \frac{\partial r}{\partial x} = \frac{x}{r} = \cos\theta$

$\frac{\partial \theta}{\partial x}$ : $\frac{\partial y}{\partial x} = 0 = \frac{\partial (r\sin\theta)}{\partial x} = \sin\theta\frac{\partial r}{\partial x} + r\frac{\partial \sin\theta}{\partial x}$

$0 = \sin\theta\cos\theta + r\cos\theta\frac{\partial \theta}{\partial x}$

$\frac{\partial \theta}{\partial x} = -\frac{\sin\theta}{r}$

w.r.t $y$: $\dfrac{\partial r}{\partial y} = \dfrac{y}{r} = \sin\theta$

$$\frac{\partial \theta}{\partial y} = \frac{x}{r^2} = \frac{\cos\theta}{r}$$

Chain rule :-

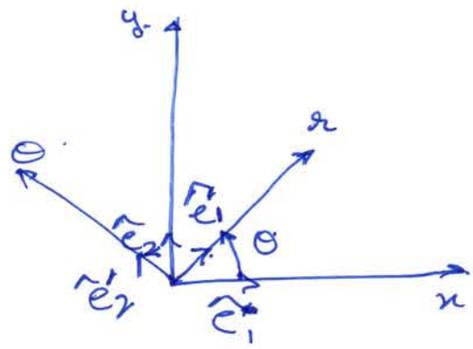
$$\frac{\partial}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}$$

$$\frac{\partial}{\partial y} = \frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial y}\frac{\partial}{\partial \theta} = \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}$$

Displacements:

$$\beta_{ij} = \hat{e}'_i \cdot \hat{e}_j$$

$$[\beta] = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$




$$\hat{u}' = \begin{Bmatrix} u_r \\ u_\theta \\ u_z \end{Bmatrix} = [\beta] \begin{Bmatrix} u_x \\ u_y \\ u_z \end{Bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} u_x \\ u_y \\ u_z \end{Bmatrix}$$

i.e- $u_r = u_x\cos\theta + u_y\sin\theta$
$u_\theta = -u_x\sin\theta + u_y\cos\theta$
$u_z = u_z$

$u_x = u_r\cos\theta - u_\theta\sin\theta$
$u_y = u_r\sin\theta + u_\theta\cos\theta$
$u_z = u_z$

OR.

Strains:

$$[\epsilon'] = [B][\epsilon][B^T]$$

$$[\epsilon] = [B]^T[\epsilon'][B]$$

i.e. $$\epsilon_{xx} = \epsilon_{rr}\cos^2\theta + \epsilon_{\theta\theta}\sin^2\theta - \epsilon_{r\theta}\sin 2\theta \quad (1)$$

$$\epsilon_{yy} = \epsilon_{rr}\sin^2\theta + \epsilon_{\theta\theta}\cos^2\theta + \epsilon_{r\theta}\sin 2\theta$$

$$\epsilon_{xy} = (\epsilon_{\theta\theta} - \epsilon_{rr})\cos\theta\sin\theta + \epsilon_{r\theta}(\cos^2\theta - \sin^2\theta)$$

$$\epsilon_{yz} = -\epsilon_{rz}\sin\theta + \epsilon_{z\theta}\cos\theta$$

$$\epsilon_{xz} = \epsilon_{rz}\cos\theta + \epsilon_{z\theta}\sin\theta$$

$$\epsilon_{zz} = \epsilon_{zz}$$

But :- $$\epsilon_{xx} = \frac{\partial u_x}{\partial x} = \frac{\partial}{\partial x}(\cos\theta\, u_r - \sin\theta\, u_\theta)$$

$$= \left(\cos\theta\frac{\partial}{\partial r} - \frac{1}{r}\sin\theta\frac{\partial}{\partial\theta}\right)(\cos\theta\, u_r - \sin\theta\, u_\theta)$$

$$\epsilon_{xx} = \cos^2\theta\frac{\partial u_r}{\partial r} + \frac{\sin^2\theta}{r}u_r - \frac{\sin\theta\cos\theta}{r}\frac{\partial u_r}{\partial\theta}$$

$$+ \frac{\sin^2\theta}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{\cos\theta\sin\theta}{r}u_\theta \quad (2).$$

$$- \frac{\cos\theta\sin\theta}{r}\frac{\partial u_\theta}{\partial r}$$

Compare (1) & (2):

$$\epsilon_{rr} = \frac{\partial u_r}{\partial r} \quad ; \quad \epsilon_{\theta\theta} = \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta}$$

$$\epsilon_{zz} = \frac{\partial u_z}{\partial z}$$

$$\epsilon_{r\theta} = \frac{1}{2}\left[\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right]$$

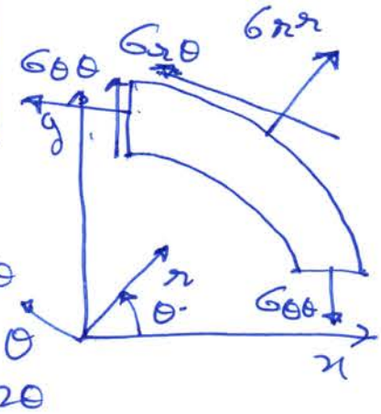
$$\epsilon_{rz} = \frac{1}{2}\left[\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right]$$

$$\epsilon_{\theta z} = \frac{1}{2}\left[\frac{1}{r}\frac{\partial u_z}{\partial \theta} + \frac{\partial u_\theta}{\partial z}\right]$$

} strain disp relation in $r, \theta, z$ system.

— stress :

$$[\sigma_{ij}'] = \begin{bmatrix} \sigma_{rr} & \sigma_{r\theta} & \sigma_{rz} \\ \sigma_{\theta r} & \sigma_{\theta\theta} & \sigma_{\theta z} \\ \sigma_{zr} & \sigma_{z\theta} & \sigma_{zz} \end{bmatrix}$$




We get :-

$$\sigma_{xx} = \sigma_{rr}\cos^2\theta + \sigma_{\theta\theta}\sin^2\theta - \sigma_{r\theta}\sin 2\theta$$

$$\sigma_{yy} = \sigma_{rr}\sin^2\theta + \sigma_{\theta\theta}\cos^2\theta + \sigma_{r\theta}\sin 2\theta$$

$$\sigma_{zz} = \sigma_{zz}$$

$$\sigma_{xy} = (\sigma_{\theta\theta} - \sigma_{rr})\sin\theta\cos\theta + \sigma_{r\theta}(\cos^2\theta - \sin^2\theta)$$

$$\sigma_{xz} = \sigma_{rz}\cos\theta - \sigma_{\theta z}\sin\theta$$

$$\sigma_{yz} = \sigma_{rz}\sin\theta + \sigma_{\theta z}\cos\theta$$

Equilibrium:

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} + \rho b_r = 0$$

$$\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \sigma_{\theta z}}{\partial z} + \frac{2}{r}\sigma_{r\theta} + \rho b_\theta = 0$$

$$\frac{\partial \sigma_{rz}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta z}}{\partial \theta} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{1}{r}\sigma_{rz} + \rho b_z = 0.$$

- Airy stress formulation:

$$\sigma_{rr} = \sigma_{xx}\cos^2\theta + \sigma_{yy}\sin^2\theta + \sigma_{xy}\sin 2\theta$$

$$= \frac{\partial^2\phi}{\partial y^2}\cos^2\theta + \frac{\partial^2\phi}{\partial x^2}\sin^2\theta - \frac{\partial^2\phi}{\partial x \partial y}\sin 2\theta$$

$$\frac{\partial}{\partial x} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}$$

$$\frac{\partial}{\partial y} = \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}$$

Result:-

$$\sigma_{rr} = \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial \theta^2} + V \qquad ; \rho b_r = -\frac{\partial V}{\partial r}$$

$$\sigma_{\theta\theta} = \frac{\partial^2\phi}{\partial r^2} + V \qquad \rho b_\theta = \frac{-1}{r}\frac{\partial V}{\partial \theta}$$

$$\sigma_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \phi}{\partial \theta}\right)$$

Compatibility :-

$$\nabla^4 \phi = 0 \quad \text{or} \quad \nabla^2 \nabla^2 \phi = 0$$

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

if $V \neq 0$

$$\nabla^2 \nabla^2 \phi = -2\,\frac{k-1}{k+1}\,\nabla^2 V$$

— Summary given on p.g. 21.

— Procedure to obtain displacements from $\phi$.

Example :- $\phi = r \log r \cos\theta$

$$\sigma_{rr} = \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} = \frac{\cos\theta}{r}$$

$$\sigma_{\theta\theta} = \frac{\partial^2 \phi}{\partial r^2} = \frac{\cos\theta}{r}$$

$$\sigma_{r\theta} = \frac{\sin\theta}{r}$$

From strain - stress relationship (p.g. 22).

$$4\epsilon_{rr} = \frac{4 \cdot 1}{2\mu}\left[\sigma_{rr} - \frac{3-k}{4}(\sigma_{rr} + \sigma_{\theta\theta})\right]$$

i.e. $$4\mu \frac{\partial u_r}{\partial r} = 2\sigma_{rr} - \frac{3-k}{2}(\sigma_{rr} + \sigma_{\theta\theta}) \quad -(1)$$

$\epsilon_{\theta\theta}$ : $$4\mu \frac{1}{r}\left(\frac{\partial u_\theta}{\partial \theta} + u_r\right) = 2\sigma_{\theta\theta} - \frac{3-k}{2}(\sigma_{rr} + \sigma_{\theta\theta}) \quad -(2)$$

$\epsilon_{r\theta}$ : $$4\mu\left[r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) + \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right] = 4\sigma_{r\theta} \quad — (3)$$

$$4\mu \frac{\partial u_r}{\partial r} = (\kappa-1)\frac{\cos\theta}{r} \qquad (1')$$

$$4\mu \frac{1}{r}\left(\frac{\partial u_\theta}{\partial \theta} + u_r\right) = (\kappa-1)\frac{\cos\theta}{r} \qquad (2')$$

$$4\mu\left[r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) + \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right] = \frac{4}{r}\sin\theta \qquad (3')$$

Integrate $(1')$:

$$4\mu u_r = (\kappa-1)\log r\cos\theta + f'(\theta) \qquad (4)$$

Integrate $(2')$:-

$$4\mu u_\theta = (\kappa-1)\sin\theta - (\kappa-1)\log r\sin\theta + f(\theta) + g(r) \qquad (5).$$

Plug (4) & (5) into $(3')$.

$$f''(\theta) + f(\theta) - 2(\kappa+1)\sin\theta = -r\,g'(r) + g(r) = A$$

To satisfy this, we require:-

$$f''(\theta) + f(\theta) - 2(\kappa+1)\sin\theta\} = A$$

$$-r\,g'(r) + g(r) = A$$

$$\Rightarrow f(\theta) = C_1\cos\theta + C_2\sin\theta + A + 2(\kappa+1)\theta\cos\theta$$

$$g(r) = A + C_3 r$$

Then :-

$$4\mu u_r = (\kappa+1)\theta\sin\theta + (\kappa-1)\log r\cos\theta - \cos\theta + (C_2-\kappa)\cos\theta - C_1\sin\theta$$

$$4\mu u_\theta = (\kappa+1)\theta\cos\theta - (\kappa-1)\log r\sin\theta - \sin\theta - (C_2+\kappa)\sin\theta - C_1\cos\theta + C_3 r.$$

Airy stress function : $\phi = \phi(x, y)$.

such that

in $x-y-z$ :

$$\sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2} + V \qquad \rho b_x = -\frac{\partial V}{\partial x}$$

$$\sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2} + V \qquad \rho b_y = -\frac{\partial V}{\partial y}.$$

$$\sigma_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y}$$

in $r-\theta-z$ :

$$\sigma_{rr} = \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} + V \qquad \rho b_r = -\frac{\partial V}{\partial r}$$

$$\sigma_{\theta\theta} = \frac{\partial^2 \phi}{\partial r^2} + V \qquad \rho b_\theta = -\frac{\partial V}{\partial \theta}.$$

$$\sigma_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \phi}{\partial \theta}\right)$$

then $\phi$ must satisfy:

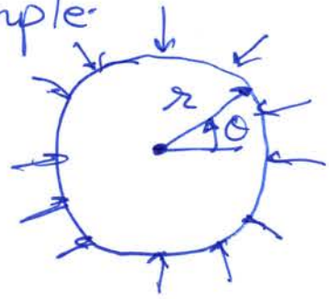
$$\nabla^4 \phi = \nabla^2 \nabla^2 \phi = -2\,\frac{\kappa - 1}{\kappa + 1}\,\nabla^2 V$$

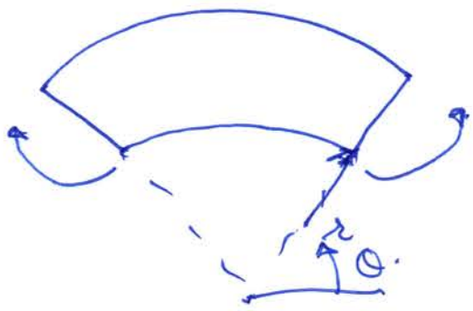
where : $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ ; $\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$

- Axisymmetric plane problems:

Axisymmetric — symmetric about an axis.

Example:




- Symmetric about z-axis.

$\phi$ is independent of $\theta$.

stresses are independent of $\theta$.

$$\sigma_{rr} = f(r) \quad , \quad \sigma_{\theta\theta} = g(r) \quad , \quad \sigma_{r\theta} = 0$$

* Displacements are not independent of $\theta$.
i.e. they are not axisymmetric.

In polar co-ordinates:

Biharmonic equation: $\nabla^4 \phi = \nabla^2 \nabla^2 \phi = 0$

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \cancelto{0}{\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}}$$

we get:-

$$\nabla^4 \phi = \frac{d^4\phi}{dr^4} + \frac{2}{r}\frac{d^3\phi}{dr^3} - \frac{1}{r^2}\frac{d^2\phi}{dr^2} + \frac{1}{r^3}\frac{d\phi}{dr} = 0$$

Convert this equation into an O.D.E with constant coefficients :

Substitute : $r = e^t$ , $t \rightarrow$ new variable

then we get :-

$$\frac{d^4\phi}{dt^4} - 4\frac{d^3\phi}{dt^3} + 4\frac{d^2\phi}{dt^2} = 0$$

General sol$^n$ :-

$$\phi = At + Bte^{2t} + Ce^{2t} + D$$

$$\boxed{\phi = A\log_n r + Br^2\log_n r + Cr^2 + D}$$

$\phi$ is the general solution for an axisymmetric problem.

$\phi$ gives :-

$$\sigma_{rr} = \frac{1}{r}\frac{\partial\phi}{\partial r} = \frac{A}{r^2} + B(1 + 2\log r) + 2C$$

$$\sigma_{\theta\theta} = \frac{d^2\phi}{dr^2} = -\frac{A}{r^2} + B(3 + 2\log r) + 2C$$

$$\sigma_{r\theta} = \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right) = 0$$

To get displacements from stresses

$$\epsilon_{rr} = \frac{\partial u_r}{\partial r} = \frac{1}{2\mu}\left[\sigma_{rr} - \frac{3-K}{4}(\sigma_{rr} + \sigma_{\theta\theta})\right]$$

$$= \frac{1}{2\mu}\left[\left(1 - \frac{3-K}{4}\right)\sigma_{rr} - \frac{3-K}{4}\sigma_{\theta\theta}\right]$$

Let $\frac{3-K}{4} = \bar{\nu}$

$$\epsilon_{rr} = \frac{\partial u_r}{\partial r} = \frac{1}{2\mu}\left[(1-\bar{\nu})\sigma_{rr} - \bar{\nu}\sigma_{\theta\theta}\right]$$

i.e. $$\frac{\partial u_r}{\partial r} = \frac{1}{2\mu}\left[\frac{A}{r^2} + B\{1 - 4\bar{\nu} + 2(1-2\bar{\nu})\log r\} + 2(1-2\bar{\nu})C\right]$$

Integrate:

$$2\mu u_r = \frac{-A}{r} + Br\left[2\log r - 4\bar{\nu}\log r - 1\right] + 2(1-2\bar{\nu})Cr + g_1(\theta)$$

Similarly:

$$\epsilon_{\theta\theta} = \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} = \frac{1}{2\mu}\left[\frac{-A}{r^2} + B\{3 - 4\bar{\nu} + 2(1-2\bar{\nu})\log r\} + 2(1-2\bar{\nu})C\right]$$

Simplify:-

$$\frac{2\mu}{r}\frac{\partial u_\theta}{\partial \theta} = 4B(1-\bar{\nu}) - \frac{1}{r}g_1'(\theta)$$

i.e. $$\frac{2\mu u_\theta}{r} = 4B(1-\bar{\nu})\theta - \frac{1}{r}g_1(\theta) + G(r)$$

i.e. $2\mu u_\theta = 4B(1-\bar{\nu})r\theta - g_1(\theta) + g_2(r)$

$g_1(\theta)$ & $g_2(r)$ are functions of integration

$$2\epsilon_{r\theta} = \frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} = \frac{1}{\mu}\sigma_{r\theta} = 0$$

$$\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} = 0$$

i.e. $g_1''(\theta) + g_1(\theta) + r g_2'(r) - g_2(r) = 0$

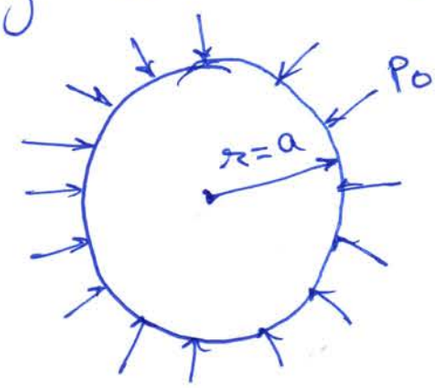
i.e. $g_1''(\theta) + g_1(\theta) = g_2(r) - r g_2'(r) = \lambda$

i.e. $g_1(\theta) = C_1\cos\theta + C_2\sin\theta + \lambda$

& $g_2(r) = C_3 r + \lambda$

---

Example:- Uniform pressure acting on a solid cylidder.



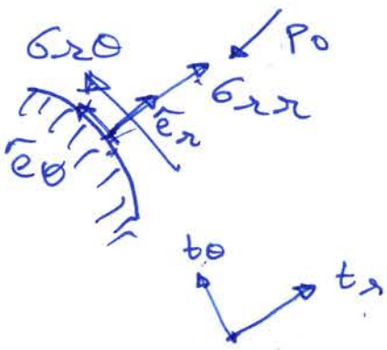

→ Axisymmetric problem:-

$\phi = A\log r + Br^2\log r + Cr^2$

$\sigma_{rr} = \frac{A}{r^2} + B(1+2\log r) + 2C$

$\sigma_{\theta\theta} = -\frac{A}{r^2} + B(3+2\log r) + 2C$

$\sigma_{r\theta} = 0$




at $r = a$.

$$t_r = \sigma_{rr}\hat{n}_r + \sigma_{r\theta}\hat{n}_\theta = -P_0$$

$$= \sigma_{rr}(1) + \sigma_{r\theta}(0) = -P_0$$

i.e. $\sigma_{rr} = -P_0$ ——— (1)

$$t_\theta = \sigma_{r\theta}\hat{n}_r + \sigma_{\theta\theta}\hat{n}_\theta = 0$$

$$= \sigma_{r\theta} = 0$$

– Solid cylinder : at $r = 0$ :-

$\sigma_{rr}$ = undefined, = $\sigma_{\theta\theta}$

due to terms $\frac{A}{r^2}$ , $B(1 + 2\log r)$

↑ infinite ; ↑ not defined.

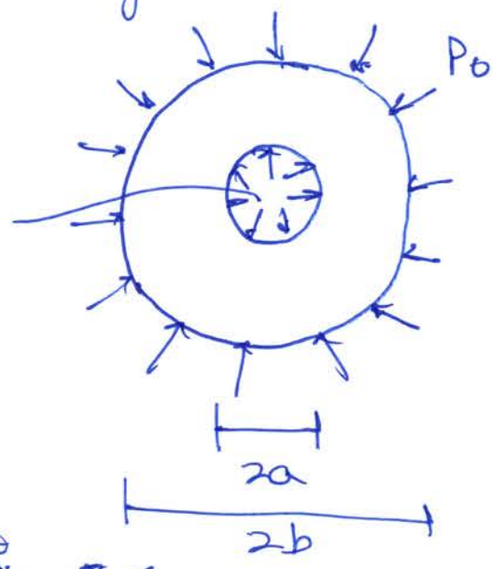
Thus, A & B must be zero.

$\therefore \sigma_{rr} = -P_0 = 2C \Rightarrow C = -\frac{P_0}{2}$

$\sigma_{\theta\theta} = 2C = -P_0$

$\therefore \phi = -\frac{P_0}{2} r^2$

Example :- Pressurized hollow cylinder.




- Axisymmetric prob:

$$\sigma_{rr} = \frac{A}{r^2} + B[1 + 2\log r] + 2C$$

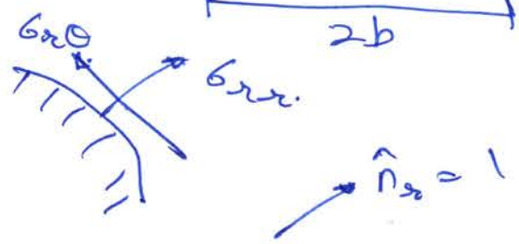
$$\sigma_{\theta\theta} = -\frac{A}{r^2} + B[3 + 2\log r] + 2C$$

$$\sigma_{r\theta} = 0$$

B.C. :

at $r = b$ : $t_r = \sigma_{rr} = -P_o$

$$t_\theta = \sigma_{r\theta} = 0$$



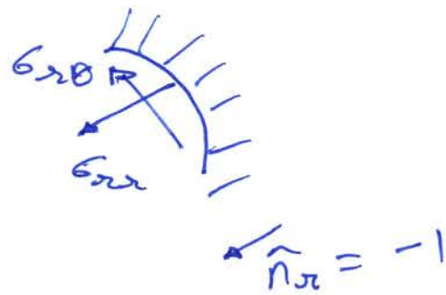

at $r = a$ :

$$t_r = \sigma_{rr}\hat{n}_r + \sigma_{r\theta}\hat{n}_\theta = P_i$$

$$= \sigma_{rr}(-1) + 0 = P_i$$

i.e. $\sigma_{rr} = -P_i$

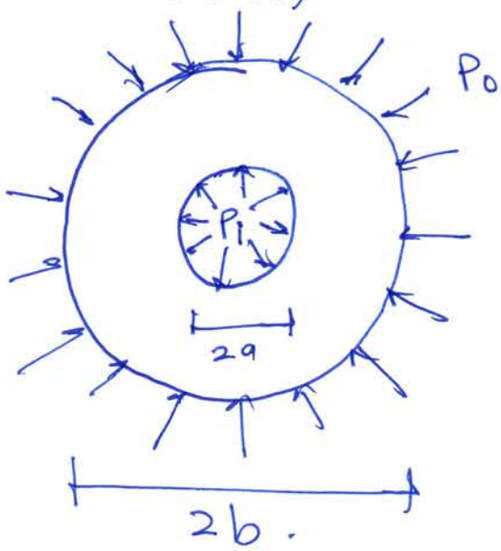
$$t_\theta = -\sigma_{r\theta} = 0$$



Example: Pressurized cylinder (Lame' sol)

Uniform external pressure $p_o$

internal pressure $p_i$




For any axisymmetric problem :-

$$\phi = A\log r + Br^2\log r + Cr^2$$

Stresses :-

$$\sigma_{rr} = \frac{A}{r^2} + B\left[1 + 2\log r\right] + 2C$$

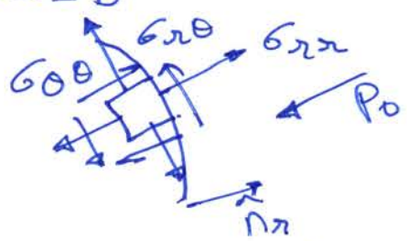
$$\sigma_{\theta\theta} = -\frac{A}{r^2} + B\left[3 + 2\log r\right] + 2C$$

$$\sigma_{r\theta} = 0$$

Find A, B & C from boundary conditions.

Boundary conditions

@ $r = b$




$\hat{n}_r = 1$, $\hat{n}_\theta = 0$

$$t_r = -P_o = \sigma_{rr}\hat{n}_r + \sigma_{r\theta}\hat{n}_\theta$$

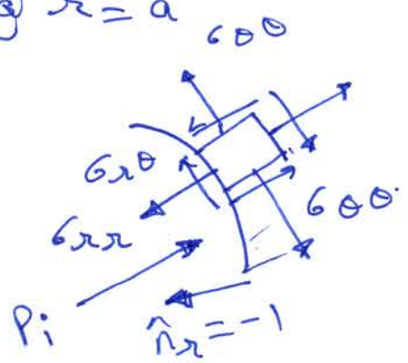
$$-P_o = \sigma_{rr}(1) + \sigma_{r\theta}(0)$$

i.e. $\boxed{\sigma_{rr} = -P_o}$ — (1)

$$t_\theta = 0 = \sigma_{r\theta}(1) + \sigma_{\theta\theta}(0)$$

$$\boxed{\sigma_{r\theta} = 0}$$

(a) $r = a$




$$t_r = P_i = \sigma_{rr}\hat{n}_r + \sigma_{r\theta}\hat{n}_\theta$$

$$P_i = \sigma_{rr}(-1) + \sigma_{r\theta}(0)$$

i.e. $\boxed{\sigma_{rr} = -P_i}$ — (2).

$$t_\theta = 0 = -\sigma_{r\theta}$$

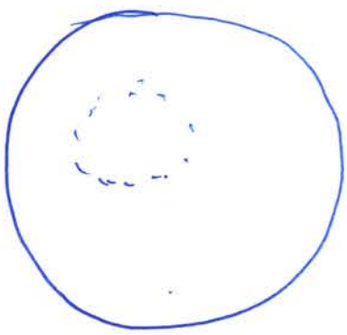
(1) at $r = b$ :

$$-P_o = \frac{A}{b^2} + B\left[1 + 2\log b\right] + 2C$$

(2) at $r = a$

$$-P_i = \frac{A}{a^2} + B\left[1 + 2\log a\right] + 2C$$

2 Eq $\longrightarrow$ 3 unknowns.



Simply connected

Multiply connected

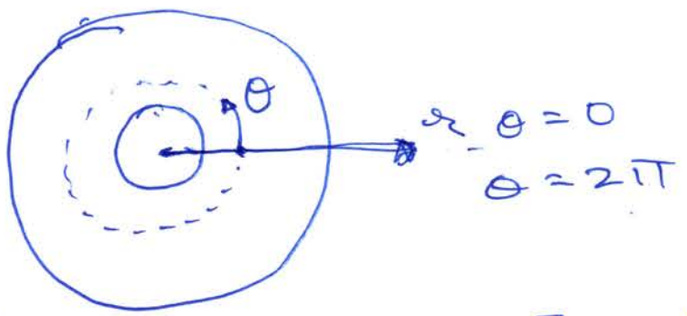
Simply connected body: "Every closed curve drawn in the body can be shrunk to a point without passing out of the body".

- Compatibility → gives us single valued displacement.

→ Compatibility is necessary and sufficient for simply connected bodies.

- Compatibility is only necessary for multiply connected bodies. But it is no longer sufficient for single valued displacements.

In this problem




$$2\mu u_r = -\frac{A}{r} + Br\left[-1 + 2\log r - 4\bar{\nu}\log r\right] + 2(1-2\bar{\nu})Cr$$
$$- C_1 \sin\theta + C_2 \cos\theta$$

$$2\mu u_\theta = 4B(1-\bar{\nu})r\theta - C_1\cos\theta - C_2\sin\theta + C_3 r + \lambda$$

We have :- $\cos 0 = \cos 2\pi$, $\sin 0 = \sin 2\pi$

$$u_r|_{\theta=0} = u_r|_{\theta=2\pi}$$

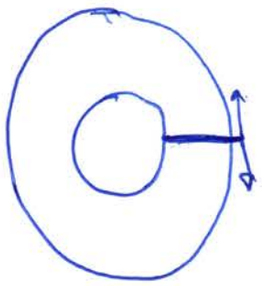
$$u_\theta|_{\theta=0} \neq u_\theta|_{\theta=2\pi}$$

$$u_\theta\Big|_{\theta=0} = -C_1 + C_3 r + \lambda$$

$$u_\theta\Big|_{\theta=2\pi} = 4\pi B(1-\nu) r (2\pi) - C_1 + C_3 r + \lambda$$

We need single valued disp.

$u_\theta\Big|_{\theta=0} = u_\theta\Big|_{\theta=2\pi}$ only if $B = 0$.



(1) $B = 0$

(2) at $r = a$ : $\frac{A}{a^2} + 2C = -P_i$

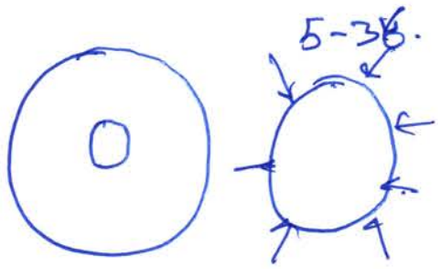
(3) at $r = b$ : $\frac{A}{b^2} + 2C = -P_o$

Solve 2 & 3 to get:-

$$A = (P_o - P_i)\frac{a^2 b^2}{b^2 - a^2}, \quad 2C = \frac{a^2 P_i - b^2 P_o}{b^2 - a^2}$$

i.e. $$\sigma_{rr} = \frac{1}{1 - \frac{a^2}{b^2}}\left[\frac{a^2}{b^2} P_i - P_o + \frac{a^2}{r^2}(P_o - P_i)\right]$$

$$\sigma_{\theta\theta} = \frac{1}{1 - \frac{a^2}{b^2}}\left[\frac{a^2}{b^2} P_i - P_o - \frac{a^2}{r^2}(P_o - P_i)\right]$$



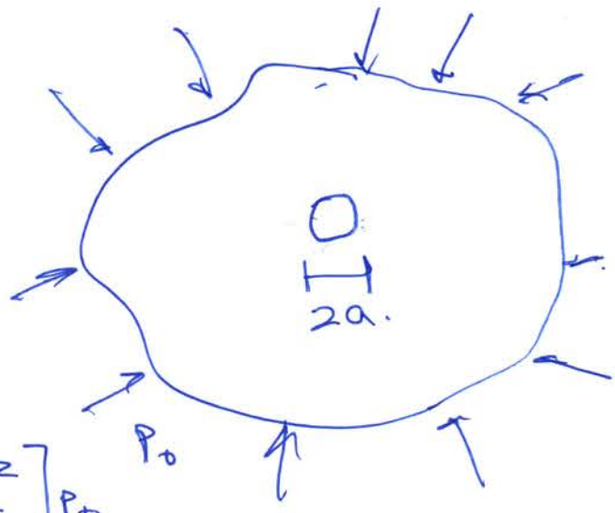
- Solid cylinder :-

$$a = 0$$

$$\sigma_{rr} = -P_o$$

$$\sigma_{\theta\theta} = -P_o .$$

- Infinite plate with a circular hole loaded remotely by $-P_o$




$$b = \infty \ , \ P_i = 0$$

$$\sigma_{rr} = -P_o + \frac{a^2}{r^2} P_o = \left[-1 + \frac{a^2}{r^2}\right] P_o$$

$$\sigma_{\theta\theta} = \left[-1 - \frac{a^2}{r^2}\right] P_o$$

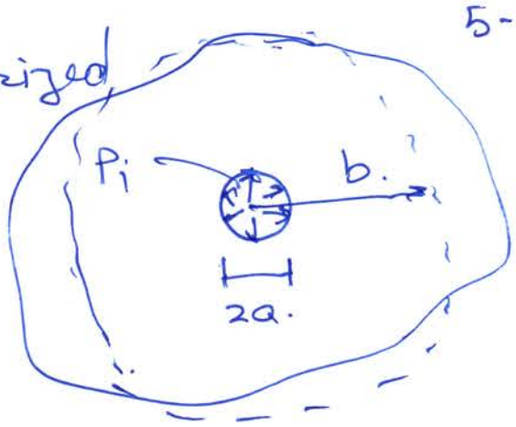
at $r = a$ :-

$$\sigma_{rr} = \left[-1 + \frac{a^2}{a^2}\right] P_o = 0$$

$$\sigma_{\theta\theta} = [-1 - 1] P_o = -2P_o$$

This is called stress concentration.

- Infinite plate with pressurized hole.




$b \to \infty$, $P_o = 0$

$$\sigma_{rr} = -\frac{a^2}{r^2} P_i$$

$$\sigma_{\theta\theta} = \frac{a^2}{r^2} P_i$$

- Fixed at $r = b$. But no tractions applied.

$$t_r\Big|_{r=b} = 0 = \sigma_{rr}\Big|_{r=b}$$

at $r = b$ : $\sigma_{rr} = 0$

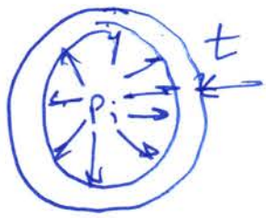
i.e. Eq (3) :- $\frac{A}{b^2} + 2C = 0$

$$\sigma_{rr}\Big|_b = 0 \quad ; \quad \sigma_{rr}\Big|_a = -P_i$$

$$\sigma_{\theta\theta}\Big|_{\substack{max \\ a}} = \frac{b^2 + a^2}{b^2 - a^2} P_i$$

i.e. $\sigma_{\theta\theta}$ increases to compensate for the fixed boundary.

Thin walled pressure vessel:




i.e. $b = a + t$

i.e. $\frac{a^2}{b^2} = \frac{(b-t)^2}{b^2}$

$= \frac{b^2 - 2bt + t^2}{b^2}$

$= 1 - \frac{2t}{b} + \left(\frac{t^2}{b^2}\right)$ ← ignore.

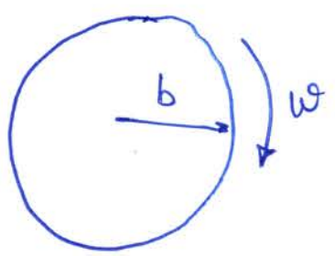
$\approx 1 - \frac{2t}{b}$

if $P_0 = 0$

then :- $\sigma_{rr} \approx 0$ , $\sigma_{\theta\theta} = P_i \frac{b}{t}$

↑ Hoop stress.

Example :- stresses in a rotating body.




Let angular velocity = $w$

Centrifugal forces will develop due to rotation.

$\rho b_r = \rho r w^2$ , $\rho b_\theta = 0$

$\rho b_r = -\frac{\partial V}{\partial r} = \rho r w^2 \Rightarrow V = -\frac{1}{2}\rho r^2 w^2$

Need to satisfy:-

$$\nabla^4 \phi = -2\frac{(K-1)}{K+1}\nabla^2 V = 4\frac{(K-1)}{K+1}\rho\omega^2$$

∵ Body force field & boundary conditions are independent of $\theta$, the prob is axisymmetric.

$$\phi = \phi_c + \phi_p$$

$$\phi = A\log r + Br^2 \log r + Cr^2 + \phi_p$$

$\phi_p \rightarrow$ Particular sol$^n$.

$$\nabla^4 \phi_p = 4\frac{(K-1)}{K+1}\rho\omega^2$$

Assume $\phi_p$ : $\bar{E} r^4 = E\left(\frac{K-1}{K+1}\rho\omega^2\right) r^4$

$$\nabla^4 \phi_p = 64\, E \frac{K-1}{K+1}\rho\omega^2 = 4\frac{(K-1)}{K+1}\rho\omega^2$$

i.e. $E = \frac{1}{16}$

i.e. $\phi_p = \frac{1}{16}\frac{K-1}{K+1}\rho\omega^2 r^4$

$$\therefore \phi = A\log r + Br^2\log r + Cr^2 + \frac{1}{16}\frac{K-1}{K+1}\rho\omega^2 r^4$$

These terms give infinite stress at $r=0$

$\therefore A = B = 0$

$$\phi = Cr^2 + \frac{1}{16}\frac{K-1}{K+1}\rho\omega^2 r^4$$

$$\sigma_{rr} = \frac{1}{r}\frac{\partial\phi}{\partial r} + V = 2C - \frac{K+3}{4(K+1)}\rho\omega^2 r^2$$

$$\sigma_{\theta\theta} = \frac{\partial^2\phi}{\partial r^2} + V = 2C + \frac{K-5}{4(K+1)}\rho\omega^2 r^2$$

$$\sigma_{r\theta} = 0$$

Boundary condition:-

at $r = b$ : $t_r = \sigma_{rr}\hat{n}_r + \sigma_{r\theta}\hat{n}_\theta$

$$t_\theta = \sigma_{r\theta}\hat{n}_r + \sigma_{\theta\theta}\hat{n}_\theta$$

$t_r = 0 \Rightarrow \sigma_{rr} = 0.$

i.e. $2C - \frac{K+3}{4(K+1)}\rho\omega^2 b^2 = 0$

i.e. $C = \frac{K+3}{8(K+1)}\rho\omega^2 b^2$

$$\phi = \frac{K+3}{8(K+1)}\rho\omega^2 b^2 r^2 + \frac{K-1}{16(K+1)}\rho\omega^2 r^4$$

$$\sigma_{rr} = \frac{K+3}{4(K+1)}\rho\omega^2\left[b^2 - r^2\right]$$

$$\sigma_{\theta\theta} = \frac{K+3}{4(K+1)}\rho\omega^2\left[b^2 + \frac{K-5}{K+3}r^2\right]$$

## 5.4 Selecting stress function

$$\nabla^4 \phi = 0.$$

Let $\phi = f(r)e^{i\theta}$ — J. H. Michell (1899).

General sol$^n$:-

$$\phi(r,\theta) = A_0 + B_0\theta + A\log r + Br^2\log r + Cr^2 + Dr^2\theta$$
$$+ \left[A_1 r + B_1 r^3 + C_1 \frac{1}{r} + D_1 r\log r + Er\theta\right] \begin{cases} \sin\theta \\ \text{or} \\ \cos\theta \end{cases}$$
$$+ \sum_{m=2}^{\infty} \left[A_m r^m + B_m r^{m+2} + C_m \frac{1}{r^m} + D_m \frac{r^2}{r^m}\right] \begin{cases} \sin m\theta \\ \\ \cos m\theta \end{cases}$$
$$+ \sum_{m=2}^{\infty} r^m \left[A_m \cos m\theta + B_m \sin m\theta + C_m \cos(m-2)\theta + D_m \sin(m-2)\theta\right]$$

Specific prob $\longleftrightarrow$ specific combination.

In x-y system:-

$\phi$ = harmonic function

i.e. $\nabla^2 \phi = 0$, then.

$x\phi$, $y\phi$ $\longrightarrow$ biharmonic.

i.e. $\nabla^4 x\phi = 0$ & $\nabla^4 y\phi = 0$.

- Analytic functions:

Def^n:- A function $f(z)$ is analytic in a region R of the complex plane if $f(z)$ has a derivative at each point in R and $f(z)$ has a single value at each point.

- If $f(z)$ is analytic at a point in z, then $f(z)$ has continuous derivatives of all orders at z.

If $z = x + iy$, $f(x,y) = u(x,y) + i\,v(x,y)$

then $f(x,y)$ is analytic if $\frac{\partial f}{\partial \bar{z}} = 0$ $\quad \bar{z} = x - iy$.

(1) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$

(2) $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

} Cauchy - Riemann equations.

Differentiate (1) w.r.t $x$ & (2) w.r.t $y$ & add:-

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial x \partial y} = 0$$

$\nabla^2 u = 0$ ← i.e. u is harmonic.

Similarly: $\nabla^2 v = 0$ i.e. v is harmonic.

i.e. Real & imaginary parts of an analytic function are harmonic functions.

Assume :- $\phi = \phi_{Real} + i\,\phi_{imag.}$

$= u(x,y) + iv(x,y)$

$= (x + iy)^n = z^n$

$z^n$ is an analytic function:

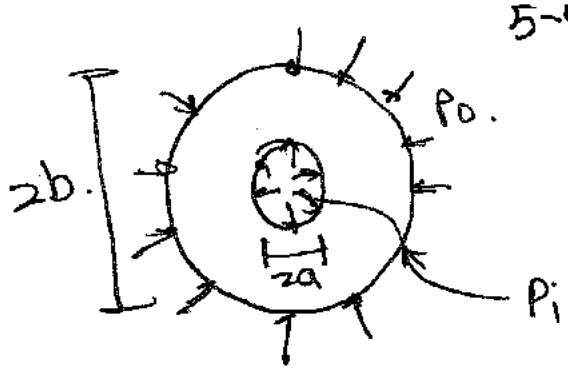
| n | $\phi_R$ | $\phi_I$ |
|---|---|---|
| 1 | $x$ | $y$ |
| 2 | $x^2 - y^2$ | $2xy$ |
| 3 | $x^3 - 3xy^2$ | $3x^2y - y^3$ |
| 4 | $x^4 - 6x^2y^2 + y^4$ | $4x^3y - 4xy^3$ |
| 5 | $x^5 - 10x^3y^2 + 5xy^4$ | $5x^4y - 10x^2y^3 + y^5$ |

Theorem :- If $\phi(x,y)$ is harmonic, then $x\phi$ and $y\phi$ are ~~at~~ biharmonic. Thus, we can use $\phi_R$ & $\phi_I$ of an analytic function to find a suitable biharmonic function.

| $x\phi$ | $y\phi$ |
|---|---|
| $x^2$ : $xy$ | $xy$ : $y^2$ |
| $x^3 - xy^2$ : $x^2y$ | $x^2y - y^3$ : $xy^2$ |
| ⋮ | ⋮ |

In general, if $\phi_0$, $\phi_1$, $\phi_2$ are any three harmonic functions, then we get a biharmonic function as $\phi(x,y) = \phi_0(x,y) + x\,\phi_1(x,y) + y\,\phi_2(x,y)$.

Example :-




B.C :-

at $r = a$ :-

$\sigma_{rr} = -P_i$ , $\sigma_{r\theta} = 0$

at $r = b$

$\sigma_{rr} = -P_o$ , $\sigma_{r\theta} = 0$

- i.e. $\sigma_{rr}$ should be constant at $r = a$ & $r = b$.

  i.e. $\sigma_{rr}$ should be independent of $\theta$.

- $u_r$, $u_\theta$ should be single-valued.

i.e. $\sigma_{rr}$ should not contain $\theta$, $\cos\theta$, $\sin n\theta$

$\phi$ Candidates :- $\phi \equiv r^2$, $\log r$, ~~$\theta$~~,

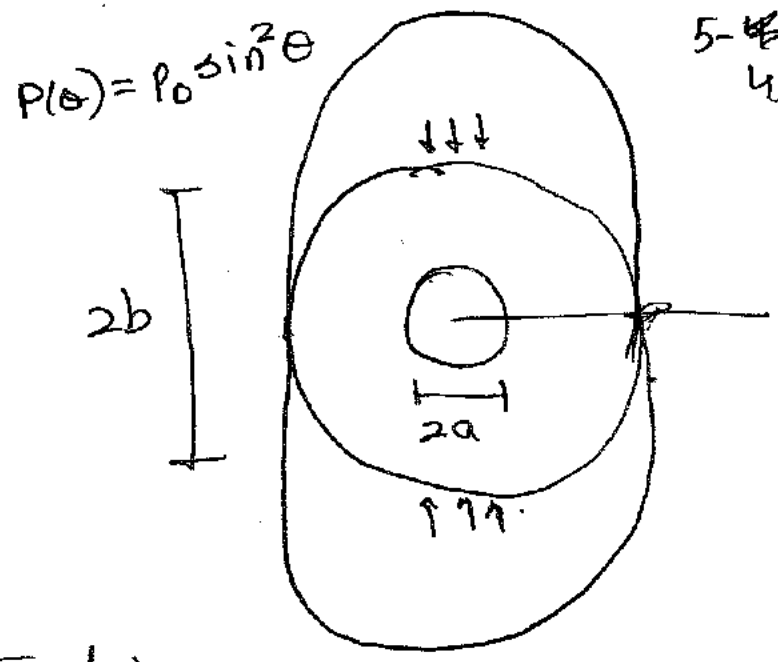
~~$r^2 \log r$~~, $r^4$, ~~$\cos 2\theta$~~,

~~$r^4 \sin 2\theta$~~.

$$\phi = Ar^2 + B\log r.$$

Example:-

Find $\phi$, $\sigma_{rr}$, $\sigma_{r\theta}$, $\sigma_{\theta\theta}$.




$$\Rightarrow P = P_0 \sin^2\theta$$

$$= P_0 \frac{(1 - \cos 2\theta)}{2}$$

$$= \frac{P_0}{2}(1 - \cos 2\theta)$$

at $r = a$ :- NO traction

$$t_r = 0 = \sigma_{rr}, \quad t_\theta = 0 = \sigma_{r\theta}$$

at $r = b$ : $t_r = P(\theta) = -\frac{P_0}{2} + \frac{P_0}{2}\cos 2\theta$

$$t_\theta = 0$$

$$t_r = \sigma_{rr}\hat{n}_r + \sigma_{r\theta}\hat{n}_\theta$$

$$t_r = \sigma_{rr} = -\frac{P_0}{2} + \frac{P_0}{2}\cos 2\theta.$$

$$t_\theta = 0 = \sigma_{r\theta}$$

at $r = a$ : $\sigma_{rr} = 0$, $\sigma_{r\theta} = 0$

$r = b$ : $\sigma_{rr} = -\frac{P_0}{2} + \frac{P_0}{2}\cos 2\theta$, $\sigma_{r\theta} = 0$.

↑ Constant (independent of $\theta$.)

↑ $\cos 2\theta$.

Select $\phi$ : $\sigma_{rr}$ must be a function of $\cos 2\theta$ or independent of $\theta$.

candidates :

| $\phi$ | | $\sigma_{r\theta}$: |
|---|---|---|
| $r^2$ | OK. | OK. |
| $\log r$. | OK | OK. |
| ~~$\theta$~~ | ~~OK~~ | ~~N.G.~~ |
| ~~$r^2 \log r$~~ : | ~~OK.~~ | ~~OK.~~ $u_\theta$ is multivalued. |
| $r^4 \cos 2\theta$ : | OK. | OK. |
| ~~$r^4 \sin 2\theta$~~ : | ~~OK~~ | ~~N.G.~~ |
| $r^2 \cos 2\theta$ : | OK. | OK. |
| $\cos 2\theta$ : | OK. | OK. |
| $\frac{\cos 2\theta}{r^2}$ : | OK. | OK. |

$$\phi = Ar^2 + B \log r + C r^4 \cos 2\theta + D r^2 \cos 2\theta + E \cos 2\theta + F \frac{\cos 2\theta}{r^2}$$

$$\sigma_{rr} = 2A + B\frac{1}{r^2} + 0 \cdot C - 2D \cos 2\theta - 4E \frac{\cos 2\theta}{r^2} - 6F \frac{\cos 2\theta}{r^4}$$

$$\sigma_{r\theta} = 0 \cdot A + 0 \cdot B + 6C r^2 \sin 2\theta + 2D \sin 2\theta - 2E \frac{\sin 2\theta}{r^2} - 6F \frac{\sin 2\theta}{r^4}$$

~~6-48~~

5-47

B.C. :-

$$\sigma_{rr}\Big|_{r=a} = 0$$

i.e. $$2A + \frac{B}{a^2} - \left[2D + \frac{4E}{a^2} + \frac{6F}{a^4}\right]\cos 2\theta = 0$$

i.e. $$2A + \frac{B}{a^2} = 0 \quad \text{—— (1)}$$

$$2D + \frac{4E}{a^2} + \frac{6F}{a^4} = 0 \quad \text{— (2)}$$

$$\sigma_{r\theta}\Big|_{r=a} = 0$$

i.e. $$\left(6Ca^2 + 2D - \frac{2E}{a^2} - \frac{6F}{a^4}\right)\sin 2\theta = 0$$

$$6Ca^2 + 2D - \frac{2E}{a^2} - \frac{6F}{a^4} = 0 \quad (3)$$

$$\sigma_{rr}\Big|_{r=b} = -\frac{P_0}{2} + \frac{P_0}{2}\cos 2\theta$$

i.e. $$\left(2A + \frac{B}{b^2}\right) - \left(2D + \frac{4E}{b^2} + \frac{6F}{b^4}\right)\cos 2\theta = -\frac{P_0}{2} + \left(\frac{P_0}{2}\right)\cos 2\theta$$

$$2A + \frac{B}{b^2} = -\frac{P_0}{2} \quad \text{—— (4)}$$

$$2D + \frac{4E}{b^2} + \frac{6F}{b^4} = -\frac{P_0}{2} \quad \text{— (5)}$$

$$\sigma_{r\theta}\big|_{r=b} = 0$$

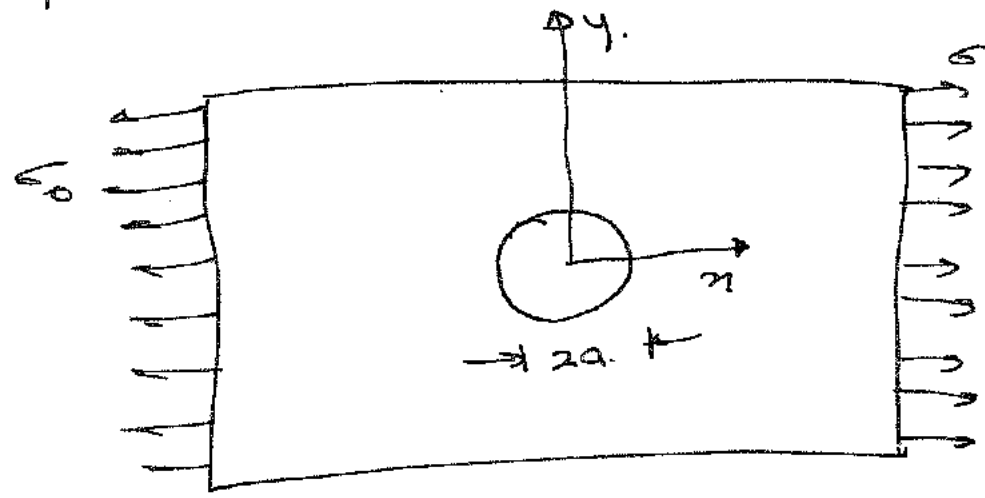
$$6Cb^2 + 2D - \frac{2E}{b^2} - \frac{6F}{b^4} = 0 \quad \text{— (6)}$$

6 Equations, 6 unknowns (A, B, C, D, E, F).

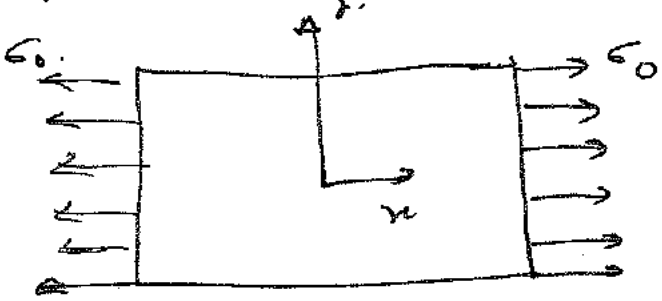
Example :- Kirsch's Problem

Hole in an infinite sheet.

Find stressed




Soln :- Problem 1.




$\sigma_{xx} = \sigma_0$

$\sigma_{yy} = \sigma_{xy} = 0.$

$$\sigma_{xx} = \frac{\partial^2 \phi_1}{\partial y^2} \Longrightarrow \phi_1 = \frac{1}{2}\sigma_0 y^2$$

Since we have a circular hole in the actual problem, we should rewrite this in cylindrical coordinates.

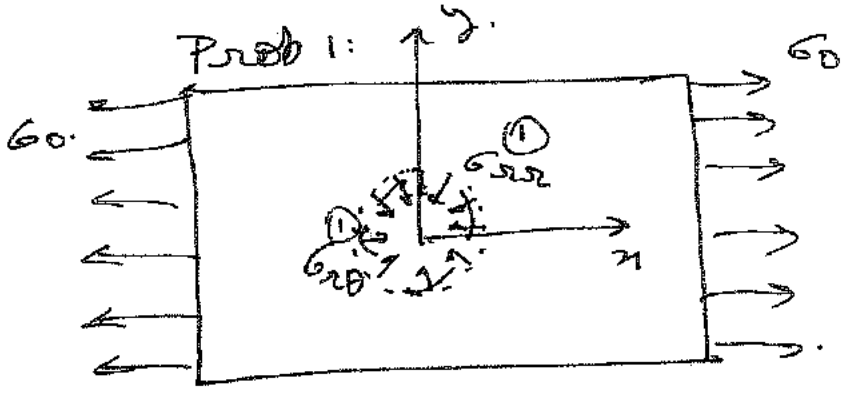
i.e. $\phi_1 = \frac{1}{2}\sigma_0 y^2 = \frac{1}{2}\sigma_0 (r\sin\theta)^2 = \frac{1}{2}\sigma_0 r^2 \sin^2\theta$

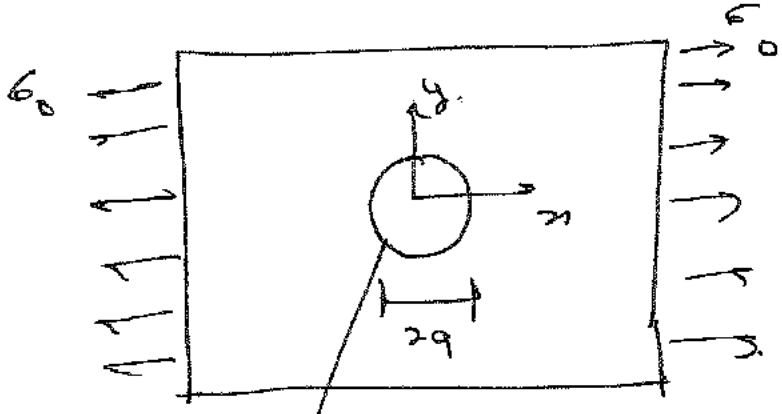
$$\phi_1 = \frac{1}{2}\sigma_0 r^2 \frac{1}{2}(1-\cos 2\theta) = \frac{1}{4}\sigma_0 r^2 - \frac{1}{4} r^2 \cos 2\theta$$

The corresponding stresses are:

$$\sigma_{rr} = \frac{1}{4}\sigma_0 2 + \frac{1}{4} 2\cos 2\theta = \frac{1}{2}\sigma_0 + \frac{1}{2}\sigma_0 \cos 2\theta$$

$$\sigma_{r\theta} = 0 - \frac{\sigma_0}{4} 2\sin 2\theta = -\frac{1}{2}\sigma_0 \sin 2\theta$$

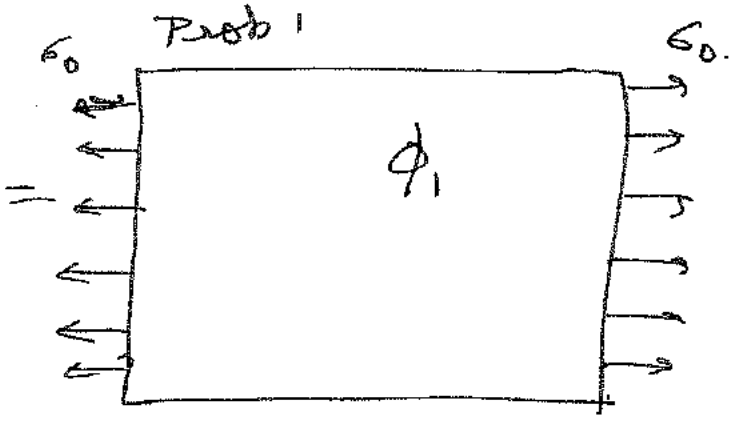


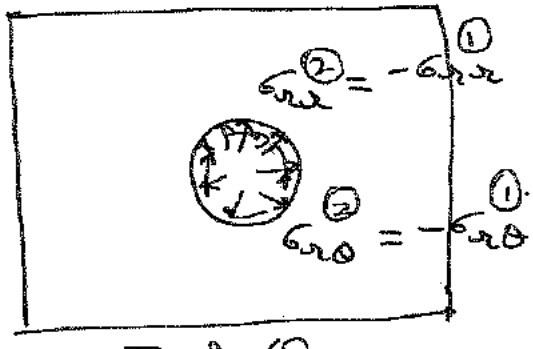

Traction free

i.e. $\sigma_{rr} = 0, \sigma_{r\theta} = 0$.

=




+




Prob (2).

i.e. find $\phi_2$ such that the hole becomes traction free while stresses at remote boundary are unchanged.

i.e. we need $\phi_2$ such that

at $r = a$ : $\sigma_{rr} = -\frac{\sigma_0}{2} - \frac{\sigma_0}{2}\cos 2\theta$

$\sigma_{r\theta} = +\frac{1}{2}\sigma_0 \sin 2\theta$.

~~re: the need~~:

From table:

Candidates

$r^2$ : No good : B.C not satisfied

$\log r$ : OK

$\theta$ : N.G.

$r^2 \log r$ : N.G. as $r \to \infty$, $\sigma_{rr} = \infty$.

$r^2\theta$ : N.G. $\sigma_{rr} \sim \theta$ not $\cos 2\theta$.

$$u_r\Big|_{\theta=0} \neq u_r\Big|_{\theta=2\pi}.$$

~~$r^2 \cos 2\theta$~~ : N.G. ~~$\sigma_{ij} \to 0$~~ $\therefore \sigma_{ij} \neq 0$ as $r \to \infty$

$\cos 2\theta$ : ✓

$\frac{\cos 2\theta}{r^2}$ : ✓

$$\therefore \quad \phi_2 = A \log r + B \frac{\cos 2\theta}{r^2} + C \cos 2\theta$$

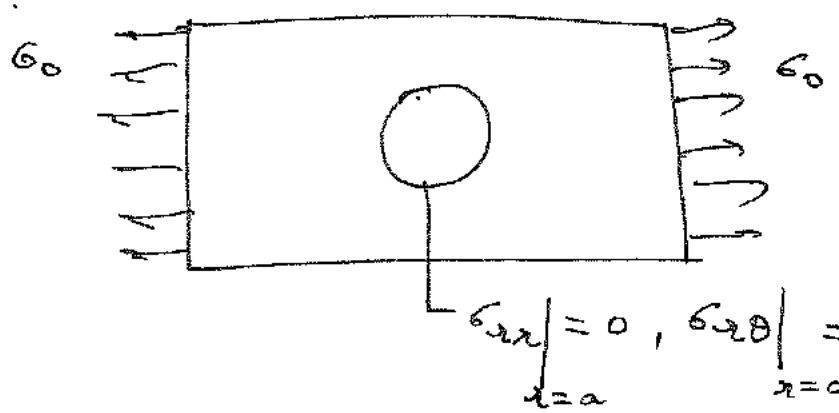
$$\therefore \quad \phi = \phi_1 + \phi_2 = \frac{1}{4}\sigma_0 \left[ r^2 - r^2 \cos 2\theta + A \log r + \frac{B}{r^2} \cos 2\theta + C \cos 2\theta \right]$$

From table:

$$\sigma_{rr} = \frac{1}{4}\sigma_0 \left[ 2 + 2\cos 2\theta + A\frac{1}{r^2} - \frac{6B}{r^4}\cos 2\theta - C\frac{4\cos 2\theta}{r^2} \right]$$

$$\sigma_{r\theta} = \frac{\sigma_0}{4} \left[ -2\sin 2\theta - B\frac{6\sin 2\theta}{r^4} - C\frac{2\sin 2\theta}{r^2} \right]$$

B.C.:




at $r=a$ : $\sigma_{rr} = 0$.

i.e. $2 + 2\cos 2\theta + \frac{A}{a^2} - B\frac{6\cos 2\theta}{a^4} - C\frac{4\cos 2\theta}{a^4} = 0$

i.e. $\left(2 + \frac{A}{a^2}\right) + \left(2 - \frac{6B}{a^4} - \frac{4C}{a^4}\right)\cos 2\theta = 0.$

i.e. $2 + \frac{A}{a^2} = 0 \qquad -(1)$

$2 - \frac{6B}{a^4} - \frac{4C}{a^4} = 0. \qquad -(2).$

at $r=b$ : $\sigma_{r\theta} = 0.$

i.e. $\left(-2 - \frac{6B}{a^4} - \frac{2C}{a^2}\right)\sin 2\theta = 0.$

i.e. $2 + \frac{6B}{a^4} + \frac{2C}{a^2} = 0. \qquad (3).$

solve (1), (2), (3) to get

$A = -2a^2, \quad B = -a^4, \quad C = 2a^2$

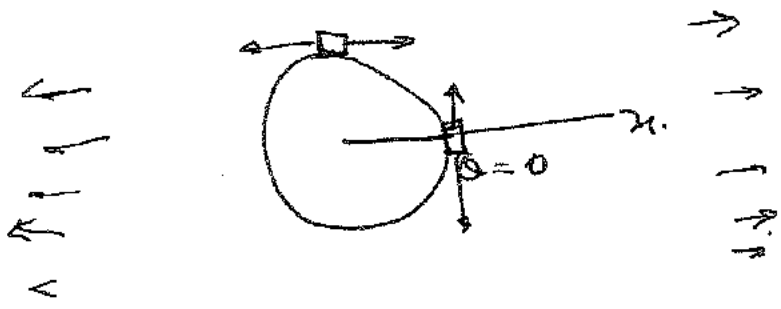
Then:

Then: $\sigma_{rr} = \frac{\sigma_0}{2}\left[1 - \frac{a^2}{r^2} + \left(1 - \frac{4a^2}{r^2} + \frac{3a^4}{r^4}\right)\cos 2\theta\right]$

$\sigma_{\theta\theta} = \frac{\sigma_0}{2}\left[1 + \frac{a^2}{r^2} - \left(1 + \frac{3a^4}{r^4}\right)\cos 2\theta\right]$

$$\sigma_{r\theta} = \frac{\sigma_0}{2}\left[-1 - \frac{2a^2}{r^2} + \frac{3a^4}{r^4}\right]\sin 2\theta.$$

$$\sigma_{\theta\theta}\Big|_{\substack{r=a,\\ \theta=0^\circ}} = -\sigma_0$$




$$\sigma_{\theta\theta}\Big|_{r=a,\ \theta=\pi/2} = 3\sigma_0.$$

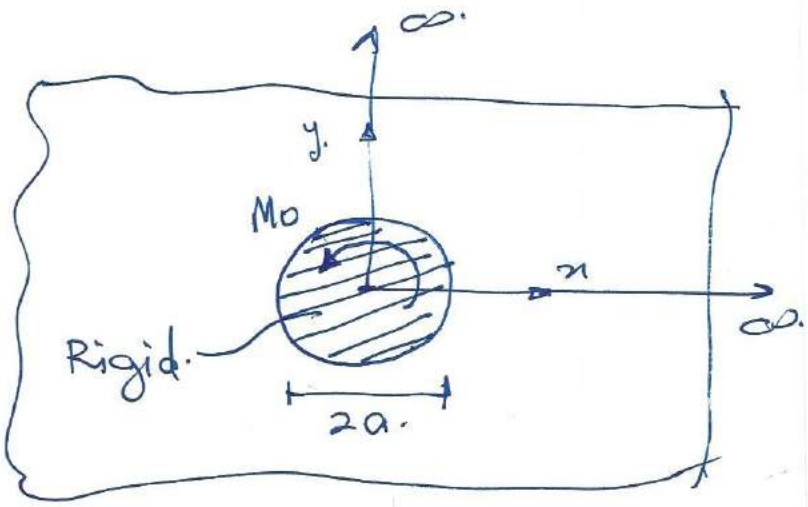
Stress concentration factor : $\dfrac{(\sigma_{\theta\theta})_{max}}{\sigma_0} = 3.$

$$\sigma_{rr}\Big|_{\theta=0} = \frac{\sigma_0}{2}\left(2 - \frac{5a^2}{r^2} + \frac{3a^4}{r^4}\right)$$

$\sigma_{rr}\Big|_{\theta=0} = \sigma_0$ at $r \to \infty$

Example:

Infinite sheet with a circular hole.
Embed a rigid disk in the hole.
Find stresses.



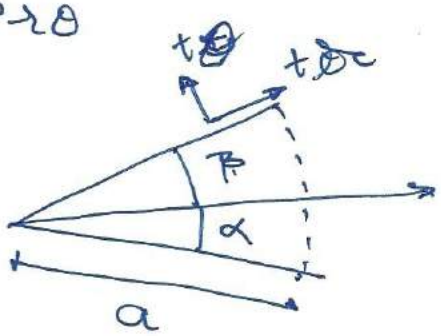

Boundary conditions:-

$$t_r = \sigma_{rr}\hat{n}_r + \sigma_{r\theta}\hat{n}_\theta \qquad \text{at } r = a.$$

$$t_\theta = \sigma_{r\theta}\hat{n}_r + \sigma_{\theta\theta}\hat{n}_\theta$$

$$t_r = \sigma_{rr}(1) + \sigma_{r\theta}(0) = \sigma_{rr} = 0$$

$$t_\theta = \sigma_{r\theta}(1) + \sigma_{\theta\theta}(0) = \sigma_{r\theta}$$




$$M = \int_{\alpha}^{\beta} a\sigma_{r\theta} \cdot a\, d\theta$$

$$\Rightarrow \quad M_0 = \int_0^{2\pi} a \cdot \sigma_{r\theta} \cdot a\, d\theta$$

$$M_0 = \int_0^{2\pi} t_\theta \cdot a^2\, d\theta.$$

Assume $t_\theta$ is constant.

$$\therefore \quad M_0 = t_\theta \cdot a^2 [\theta]_0^{2\pi} = t_\theta a^2 2\pi$$

$$\text{i.e.} \quad t_\theta = \frac{M_0}{2\pi a^2}, \quad t_r = 0$$

at $r = a$ :-

$$t_r = \sigma_{rr} = 0$$

$$t_\theta = \sigma_{r\theta} = \frac{M_0}{2\pi a^2} = \text{constant}.$$

at $r = \infty$, $\sigma_{ij} = 0$

$\sigma_{ij}$ should have a $\frac{1}{r^n}$ term, $n > 0$

Candidates:-

| | | |
|---|---|---|
| $\phi$ | | |
| $r^2$ | × | $\because \sigma_{rr} \neq 0$ at $r = a$. |
| $\log r$ | × | " " |
| $\theta$ | ✓ | |
| $r^2 \log r$ | × | $\sigma_{rr} \neq 0$ at $r = a$. |
| $r^2\theta$ | × | multivalued disp. |

$$\phi = A\theta$$

$$\sigma_{rr} = 0, \quad \sigma_{\theta\theta} = 0, \quad \sigma_{r\theta} = \frac{A}{r^2}$$

at $r = a$, $\sigma_{r\theta} = \frac{M}{2\pi a^2} = \frac{A}{a^2}$

i.e. $A = \frac{M}{2\pi}$

$$\phi = \frac{M}{2\pi}\theta \; ; \quad \sigma_{r\theta} = \frac{M}{2\pi r^2}, \quad \sigma_{rr} = \sigma_{\theta\theta} = 0$$

— Inverse Problem:-

— Flamant's Problem (1892)

$$\phi = Ax \tan^{-1}\frac{x}{y}$$

(1) Is it biharmonic? $\because \nabla^4\phi = 0$

$$\frac{\partial}{\partial x}\tan^{-1}\frac{x}{y} = \frac{y}{x^2+y^2} \qquad \frac{\partial}{\partial y}\tan^{-1}\frac{x}{y} = \frac{-x}{x^2+y^2}$$

$$\frac{\partial\phi}{\partial x} = \frac{\partial}{\partial x}\left[Ax\tan^{-1}\frac{x}{y}\right] = A\tan^{-1}\frac{x}{y} + A\frac{xy}{x^2+y^2}$$

$$= A\tan^{-1}\frac{x}{y} + A\frac{xy}{r^2}$$

$$\frac{\partial^2\phi}{\partial x^2} = \frac{2A}{r^4}y^3$$

Similarly: $\frac{\partial^2\phi}{\partial y^2} = \frac{2A}{r^4}x^2y$

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = \frac{2Ay^3}{r^4} + \frac{2A}{r^4}x^2y = \frac{2Ay}{r^2}$$

$\therefore \phi$ is not harmonic.

$$\frac{\partial^2}{\partial x^2}\nabla^2\phi = \frac{16A}{r^6}x^2y - \frac{4A}{r^4}y$$

$$\frac{\partial^2}{\partial y^2}\nabla^2\phi = -\frac{12Ay}{r^4} + \frac{16A}{r^6}y^3$$

$$\therefore \nabla^4\phi = -\frac{16Ay}{r^4} + \frac{16A}{r^6}y(x^2+y^2) = 0$$

$\phi = Ax \tan^{-1}\frac{x}{y}$ is a biharmonic function.

That is, it some boundary value problem.

(2) Stresses:-

$$\sigma_{xx} = \frac{\partial^2\phi}{\partial y^2} = \frac{2Ax^2y}{r^4}$$

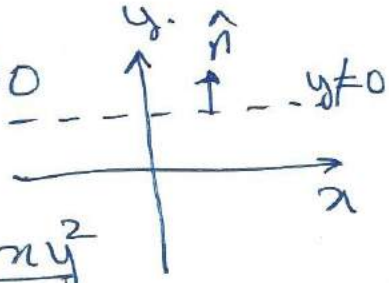
$$\sigma_{yy} = \frac{\partial^2\phi}{\partial x^2} = \frac{2Ay^3}{r^4}$$

$$\sigma_{xy} = -\frac{\partial^2\phi}{\partial x \partial y} = \frac{2Axy^2}{r^4}$$

as $r \to \infty$, $\sigma_{ij} = 0$

as $r \to 0$, $\sigma_{ij} \to \infty$

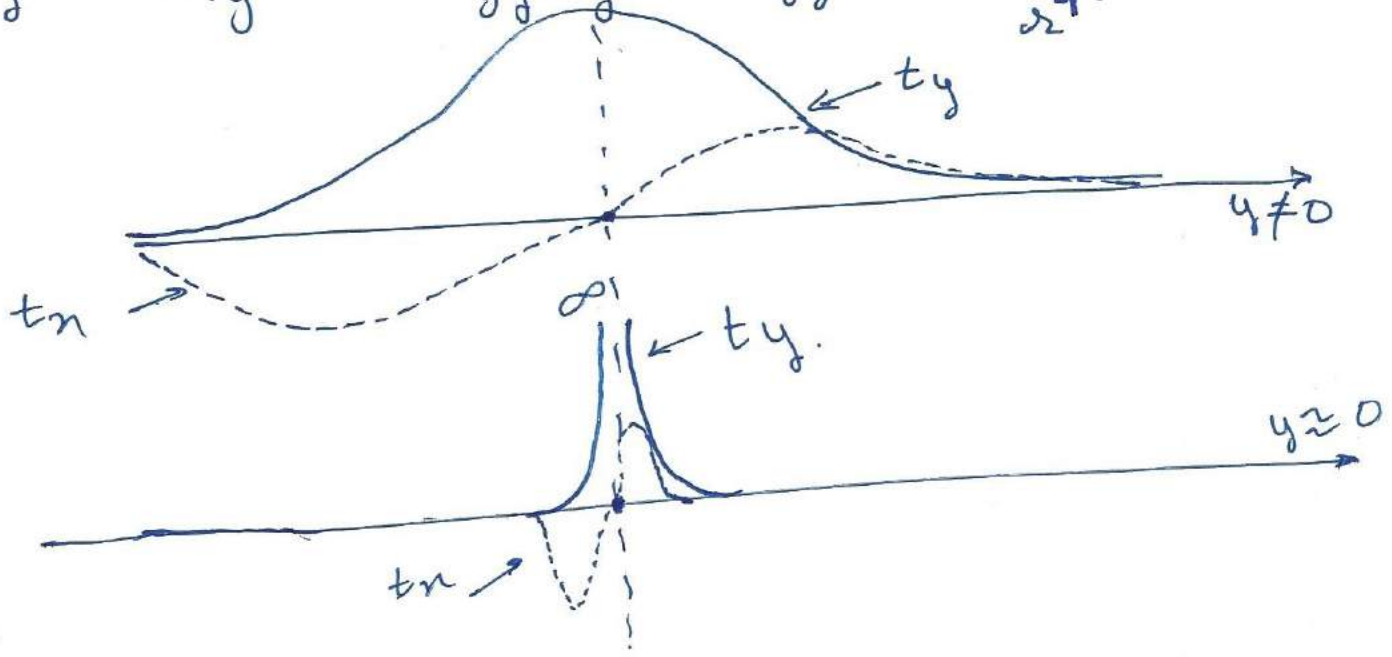
- Consider a horizontal plane: $y \neq 0$



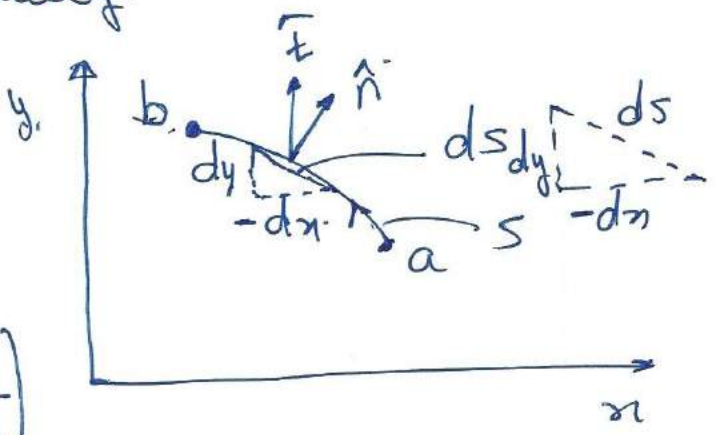

$n_x = 0$, $n_y = 1$

$\therefore t_x = \sigma_{xx}\hat{n}_x + \sigma_{xy}\hat{n}_y = \sigma_{xy} = \frac{2Axy^2}{r^4}$

$t_y = \sigma_{xy}\hat{n}_x + \sigma_{yy}\hat{n}_y = \sigma_{yy} = \frac{2Ay^3}{r^4}$



Consider a general boundary:




$$n_x = \frac{dy}{ds}, \quad n_y = \frac{-dx}{ds}$$

$$t_x = \sigma_{xx}\hat{n}_x + \sigma_{xy}\hat{n}_y$$

$$= \frac{\partial^2\phi}{\partial y^2}\frac{dy}{ds} + \frac{-\partial^2\phi}{\partial x \partial y}\left(\frac{-dx}{ds}\right)$$

$$t_x = \frac{d}{ds}\left(\frac{\partial\phi}{\partial y}\right)$$

$$t_y = \sigma_{xy}\hat{n}_x + \sigma_{yy}\hat{n}_y$$

$$= \frac{-\partial\phi}{\partial x \partial y}\frac{dy}{ds} + \frac{\partial^2\phi}{\partial x^2}\left(\frac{-dx}{ds}\right)$$

$$t_y = -\frac{d}{ds}\left(\frac{\partial\phi}{\partial x}\right)$$

Thus :-

$$F_x = \int_a^b t_x\, ds = \int_a^b \frac{d}{ds}\left(\frac{\partial\phi}{\partial y}\right) ds = \left[\frac{\partial\phi}{\partial y}\right]_a^b$$

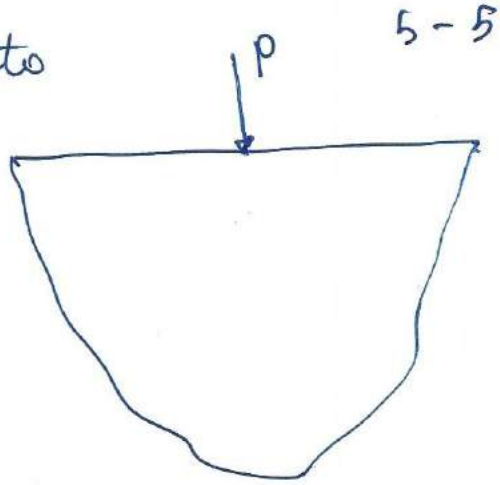
$$F_y = \int_a^b t_y\, ds = \int_a^b -\frac{d}{ds}\left(\frac{\partial\phi}{\partial x}\right) ds = -\left[\frac{\partial\phi}{\partial x}\right]_a^b$$

- On a horizontal surface $y \neq 0$ $\quad -\infty$

$$F_x = \left[\frac{\partial\phi}{\partial y}\right]_a^b = \left[\frac{\partial\phi}{\partial y}\right]_{x=\infty}^{x=-\infty} = \left[-\frac{Ax^2}{x^2}\right]_\infty^\infty = 0$$

$$F_y = -\left[\frac{\partial\phi}{\partial x}\right]_\infty^{-\infty} = -\left[A\tan^{-1}\frac{x}{y} + \frac{Axy}{x^2}\right]_{+\infty}^{\infty} = -A\pi$$

Let resultant force be equal to applied force P.




i.e. $-P = -A\pi$ i.e. $A = \frac{P}{\pi}$

$\therefore \sigma_{xx} = \frac{2P}{\pi}\frac{x^2 y}{r^4}$

$\sigma_{yy} = \frac{2P}{\pi}\frac{y^3}{r^4}$

$\sigma_{xy} = \frac{2P}{\pi}\frac{xy^2}{r^4}$

- Given :- Concentrated normal traction without any shear. ⟶ Cannot be solved using above stress function.

* Flamant's Prob :- Concentrated normal traction + shear tractions with zero resultant

Example :- $\phi(r,\theta) = Ar^2[\theta - \sin\theta\cos\theta]$

$= Ar^2[\theta - \frac{\sin 2\theta}{2}]$

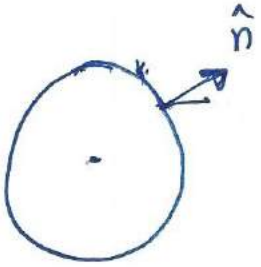
Both $r^2\theta$ & $r^2\sin 2\theta$ are in the tables.

$\therefore \phi$ is biharmonic.

$$\sigma_{rr} = A[2\theta + \sin 2\theta]$$

$$\sigma_{\theta\theta} = A[2\theta - \sin 2\theta]$$

$$\sigma_{r\theta} = A[-1 + \cos 2\theta]$$



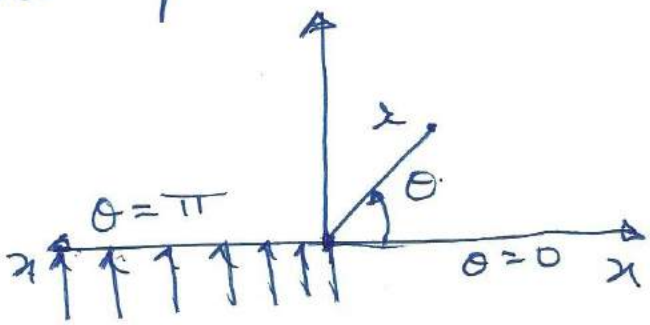

stresses are independent of $r$.

On a line $r = a$ = constant, then.

$$t_r = \sigma_{rr} n_r + \sigma_{r\theta} n_\theta = \sigma_{rr} = A[2\theta + \sin 2\theta].$$

$$t_\theta = \sigma_{r\theta} = A[-1 + \cos 2\theta]$$

$\therefore$ $t_r, t_\theta$ also independent of $r$.




For $\theta = 0$

$t_r = 0$, $t_\theta = 0$

For $\theta = \pi$

$t_r = \frac{2\pi A}{1}$ $t_\theta = 0$.

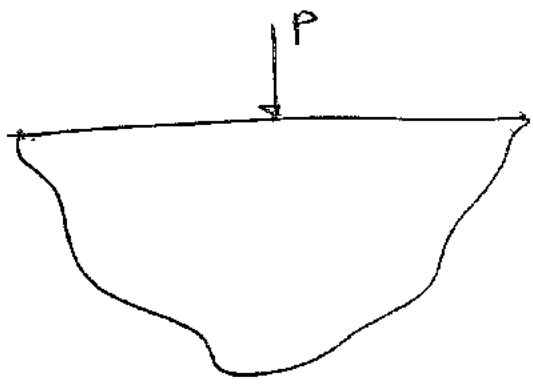
i.e. Constant normal traction on half the surface.

Normal load applied over a finite segment.

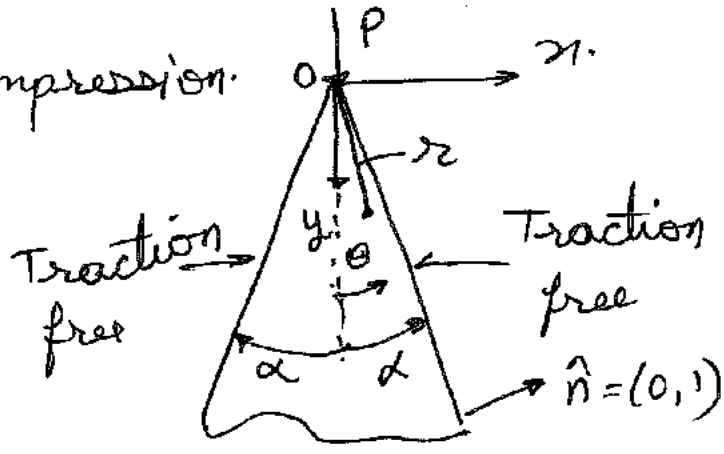
$$\phi = A\left[(x-a)^2+y^2\right]\tan^{-1}\frac{y}{x-a} - (x-a)y - \left[(x+a)^2+y^2\right]\tan^{-1}\frac{y}{x+a}$$

$$+ (x+a)y$$

~~Example :-~~




Example :- Wedge under compression.

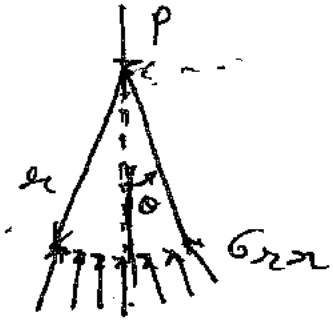



B.C :- for $\theta = \pm\alpha$

$t_r = \sigma_{r\theta} = 0$

$t_\theta = -\sigma_{\theta\theta} = 0$

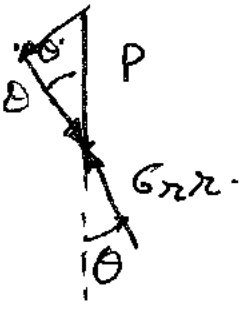
$\phi$ ?.




① $\sigma_{rr}$ generated to resist P.

@ $\theta = 0$ , $\sigma_{rr} = kP$ ↓ traction

$\sigma_{rr} = f(\cos\theta)$



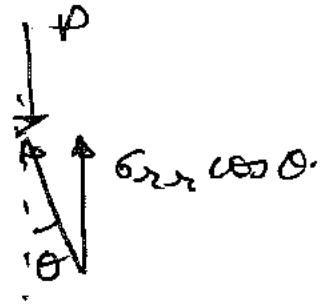

Also as $r \to \infty$, $\sigma_{rr} \to 0$

$\sigma_{rr} = f(\cos\theta)\, g\left(\frac{1}{r^n}\right)$

$\phi = Ar\theta\sin\theta$.

$\therefore \sigma_{rr} = \frac{2A\cos\theta}{r}$ , $\sigma_{r\theta} = \sigma_{\theta\theta} = 0$

Equilibrium :- $\Sigma F_y = 0$




$$\Sigma F_y = -P + \int_{-\alpha}^{\alpha} \sigma_{rr}\cos\theta\, d\theta \cdot r$$

$$\therefore \quad P = 2\int_{0}^{\alpha} \sigma_{rr}\cos\theta\, d\theta \cdot r$$

But $\sigma_{rr} = 2A\dfrac{\cos\theta}{r}$

$$\therefore \quad P = 2\int_{0}^{\alpha} 2A\frac{\cos\theta}{r}\cdot\cos\theta\, d\theta \cdot r$$

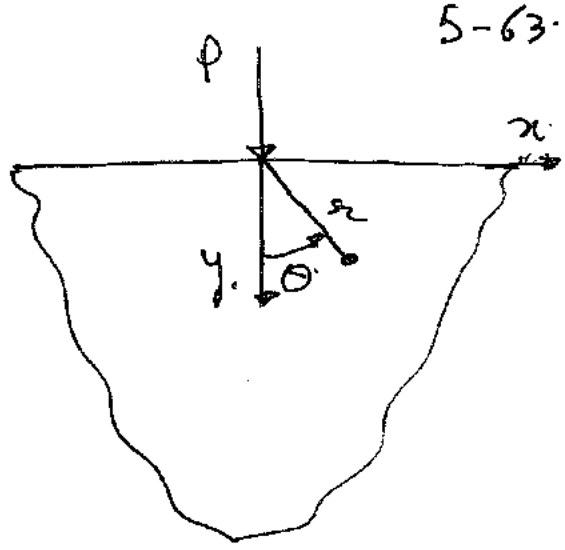
$$P = 4A\int_{0}^{\alpha} \cos^2\theta\, d\theta$$

Solve to get :- $A = \dfrac{P}{2\alpha + \frac{1}{2}\sin 2\alpha}$

$$\therefore \quad \phi = \frac{P}{2\alpha + \frac{1}{2}\sin 2\alpha}\, r\theta\sin\theta$$

$$\sigma_{rr} = \frac{P\cos\theta}{r\left(\alpha + \frac{1}{2}\sin 2\alpha\right)} \qquad \text{(compressive)}.$$

For $\alpha = \frac{\pi}{2}$




$$\phi = \frac{P}{2\pi} r\theta \sin\theta$$

$$\sigma_{rr} = \frac{2P}{\pi}\frac{\cos\theta}{r}$$

Convert into x-y system:

Transformation equations:

$$\sigma_{xx} = \sigma_{rr}\cos^2\theta + \sigma_{\theta\theta}\sin^2\theta - 2\sigma_{r\theta}\sin\theta\cos\theta$$

$$\sigma_{yy} = \sigma_{rr}\sin^2\theta + \sigma_{\theta\theta}\cos^2\theta + 2\sigma_{r\theta}\sin\theta\cos\theta.$$

$$\sigma_{xy} = (\sigma_{rr} - \sigma_{\theta\theta})\sin\theta\cos\theta + \sigma_{r\theta}(\cos^2\theta - \sin^2\theta).$$

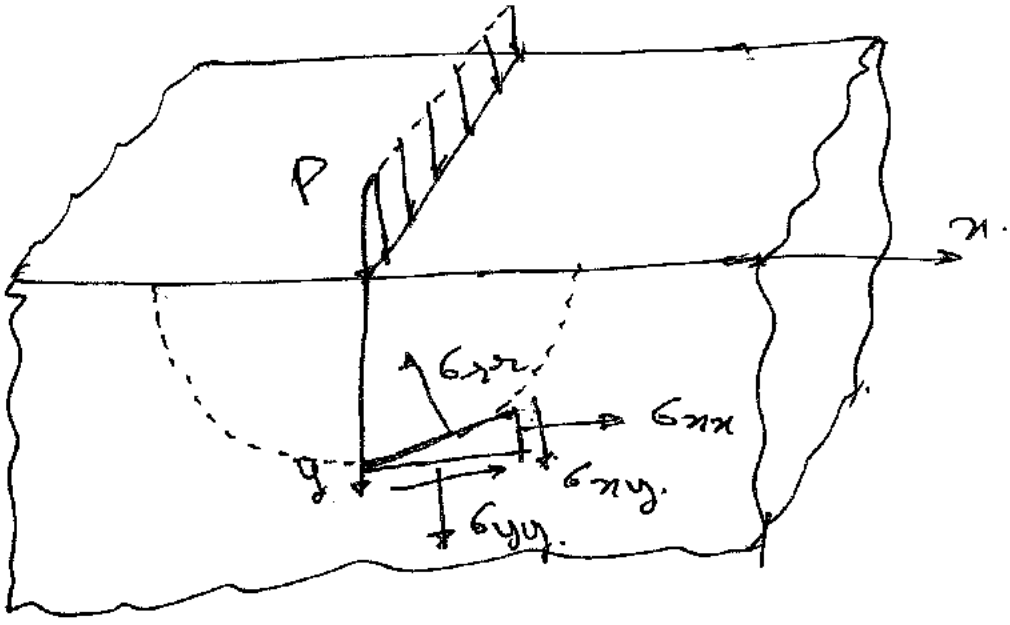
Also :- ~~$y = r\cos\theta$, $x = r\sin\theta$.~~
$y = r\sin\theta$, $x = r\cos\theta$.

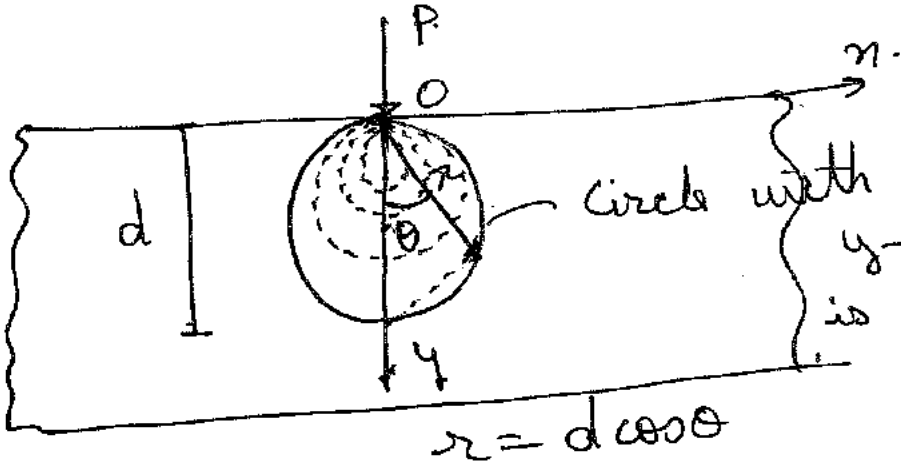
$$\sigma_{xx} = \frac{2P}{\pi}\frac{x^2 y}{(x^2+y^2)^2}$$

$$\sigma_{yy} = \frac{2P}{\pi}\frac{y^3}{(x^2+y^2)^2}$$

$$\sigma_{xy} = \frac{2P}{\pi}\frac{xy^2}{(x^2+y^2)^2}$$

} same as Flamant's sol$^n$.



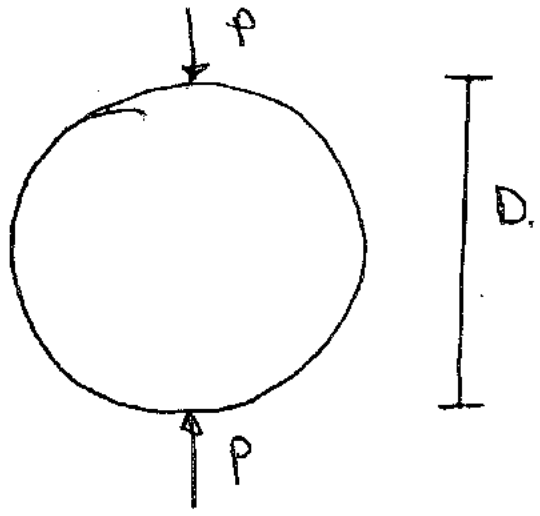




We have :- $\sigma_{rr} = \frac{2P}{\pi}\frac{\cos\theta}{r}$

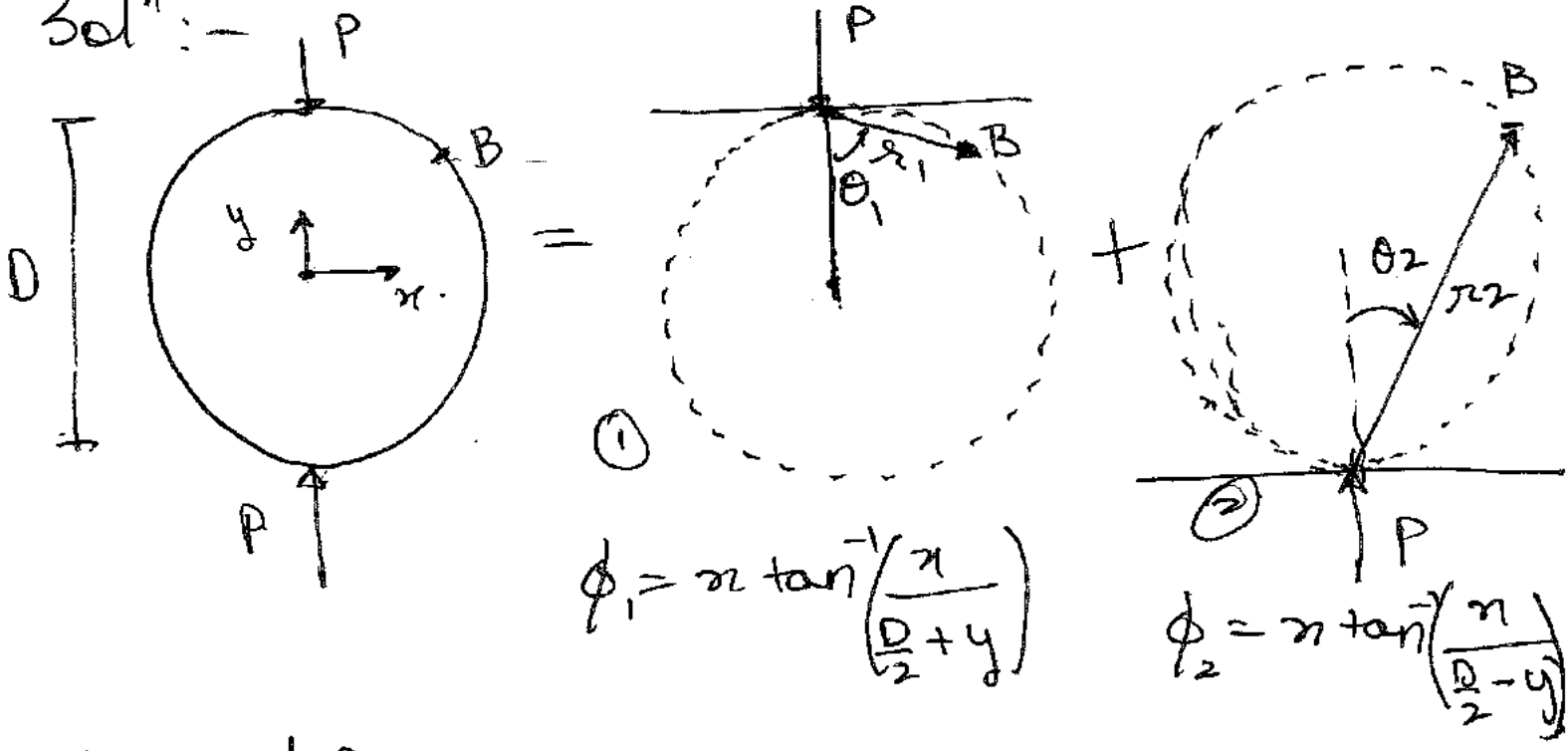
i.e- $\sigma_{rr} = \frac{2P}{\pi d}$

i.e. $\sigma_{rr}$ is same at all points on the circle, except the loading point.

Example. Disk under diametral compression.




Sol$^n$ :-




$$\phi_1 = \pi \tan^{-1}\left(\frac{x}{\frac{D}{2}+y}\right)$$

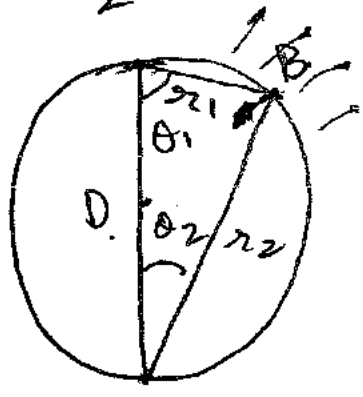
$$\phi_2 = \pi \tan^{-1}\left(\frac{x}{\frac{D}{2}-y}\right)$$

At point B,

$$\sigma_{rr}^{①} = \frac{2P}{\pi}\frac{\cos\theta_1}{r_1}, \quad \sigma_{rr}^{②} = \frac{2P}{\pi}\frac{\cos\theta_2}{r_2}$$

$$\frac{\cos\theta_1}{r_1} = \frac{\cos\theta_2}{r_2} = \frac{1}{D}$$

$$\sigma_{rr}^{①} = \frac{2P}{\pi D}, \quad \sigma_{rr}^{②} = \frac{2P}{\pi D}$$

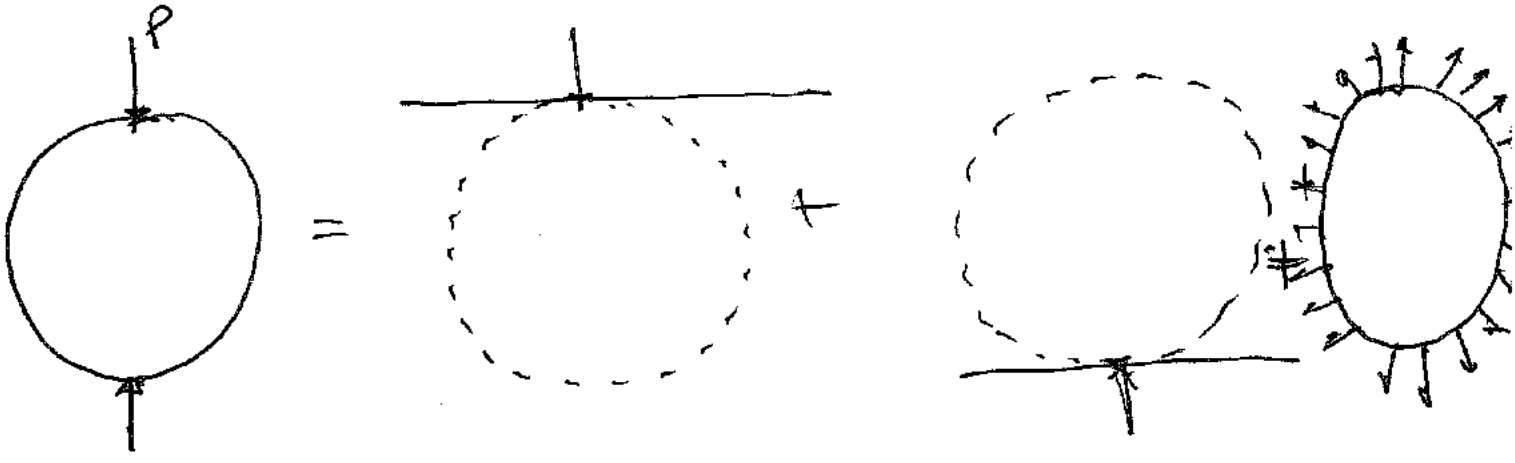


$$\sigma_{rr} = \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} \to 0.$$

$$\frac{2P}{\pi D} = \frac{1}{r}\frac{\partial \phi}{\partial r}$$

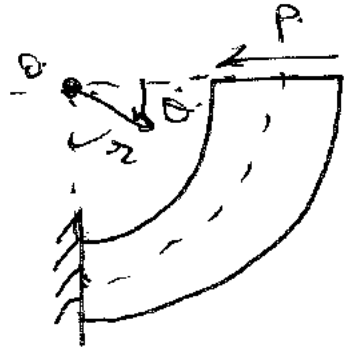
$$\phi_3 = \frac{2P\, r^2}{\pi D\, 2.} = \frac{P r^2}{\pi D} = \frac{P}{\pi D}(x^2 + y^2)$$

$\therefore$ $\phi_3 = \frac{P}{\pi D}(x^2 + y^2)$ will result in tractions required to cancel out tractions generated by Flamant's soln.

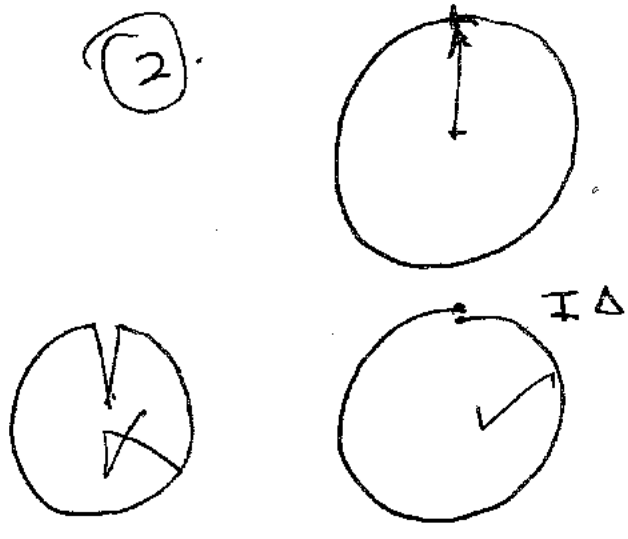



HW:-

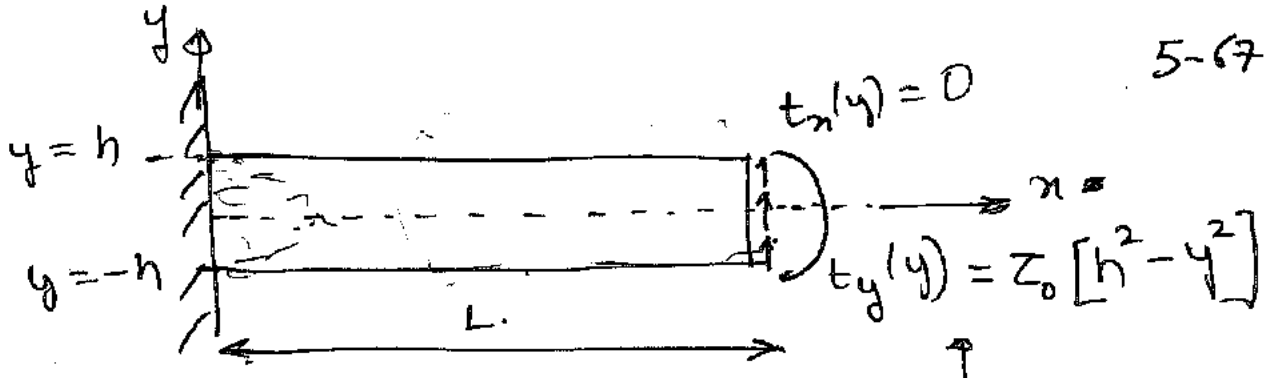
(1)




(2)



Example:




$$\phi(x,y) = Axy + Bxy^3 + Cy^3$$

Does this $\phi$ solve the above problem?

$\phi$ is biharmonic → since highest power is 3.

$$\sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2} = 6Bxy + 6Cy$$

$$\sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2} = 0$$

$$\sigma_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} = -A - 3By^2$$

Boundary conditions:-

On horizontal planes $y = \pm h$, $n_x = 0$, $n_y = \pm 1$

$$t_x = \sigma_{xx} n_x + \sigma_{xy} n_y = 0 = \sigma_{xy}$$

$$t_y = \sigma_{xy} n_x + \sigma_{yy} n_y = 0 = \sigma_{yy} \rightarrow \text{Automatically satisfied.}$$

i.e. at $y = \pm h$, $\sigma_{xy} = 0$.

i.e. $-A - 3Bh^2 = 0 \Rightarrow$ $\boxed{A = -3Bh^2}$ (1).

On face $x = L$, $n_x = 1$, $n_y = 0$.

$t_x = \sigma_{xx} = 0 = 6BLy + 6Cy$

$t_y = \tau_0[h^2 - y^2] = \sigma_{xy} = -A - 3By^2$

We have:-

$6BLy + 6Cy = 0 \Rightarrow BL + C = 0$ (2)

$\tau_0[h^2 - y^2] = -A - 3By^2$

i.e. $\boxed{A = -\tau_0 h^2}$

$\boxed{B = \frac{\tau_0}{3}}$

$\therefore A = -\tau_0 h^2$, $B = \frac{\tau_0}{3}$, $C = -BL = -\frac{\tau_0 L}{3}$

$\phi = -\tau_0 h^2 xy + \frac{\tau_0}{3} xy^3 - \frac{\tau_0}{3} Ly^3$

$\sigma_{xx} = 2\tau_0[x - L]y$

$\sigma_{yy} = 0$

$\sigma_{xy} = \tau_0[h^2 - y^2]$

} Stress field satisfies traction B.C. on 3 faces.

For this to be a complete solution, it should satisfy traction condition on all surfaces.

But we do not know the tractions at the cantilevered end. Instead, we are given $u_x = 0$, $u_y = 0$.

Thus, this is a "mixed boundary value problem".

Using strain displacement relationship :-

$$E u_x(x,y) = 2\tau_0 \left[\frac{x^2 y}{2} - xy^2\right] + f_1(y)$$

$$E u_y(x,y) = f_2(x)$$

Use shear strain - displacement eq, to find $f_1(y)$ & $f_2(x)$.

$$E u_x(x,y) = 2\tau_0 \left[\frac{x^2}{2} - xL + h^2 - \frac{y^2}{3}\right] y - \lambda y + c_1$$

$$E u_y(x,y) = \tau_0 \left[L - \frac{x}{3}\right] x^2 + \lambda x + c_2.$$

$\lambda$, $c_1$, $c_2$ are constants (rigid body motion)

We need :-

at $x=0$, $u_x = u_y = 0$

$$E u_x(0,y) = 2\tau_0 \left[h^2 - \frac{y^2}{3}\right] y - \lambda y + c_1$$

$$E u_y(0,y) = c_2$$

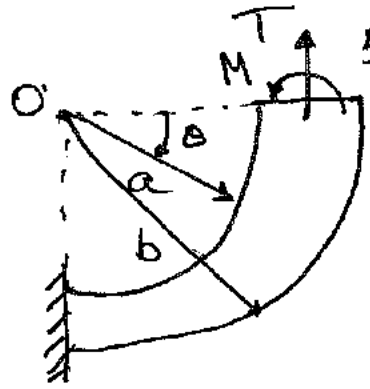
$\phi$ is not an exact solution to the given problem. But it is a approximate solution.

If we were given :- $t_x = 2\tau_0 y$ at $x=0$.

$$t_y = -\tau_0 [h^2 - y^2]$$

then $\phi$ would be exact.

Example:




$$\phi = \left[Ar^3 + \frac{B}{r} + \cancel{cr} + Dr\log r\right]\cos\theta$$

Does $\phi$ solve the problem?

If not, how will you modify $\phi$ to solve this problem?

$$\rightarrow \sigma_{rr} = \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = \left[2Ar - \frac{2B}{r^3} + \frac{D}{r}\right]\cos\theta$$

$$\sigma_{\theta\theta} = \frac{\partial^2\phi}{\partial\theta^2} = \left[6Ar + \frac{2B}{r^3} + \frac{D}{r}\right]\cos\theta$$

$$\sigma_{r\theta} = -\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right] = \left[2Ar - \frac{2B}{r^3} + \frac{D}{r}\right]\sin\theta$$

B.C. :— at $r=a$, traction free.

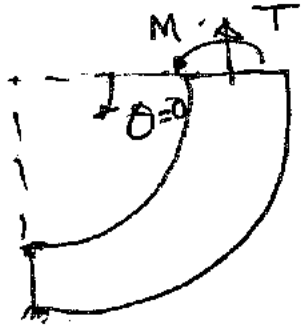
$$t_r = \sigma_{rr}n_r + \sigma_{r\theta}n_\theta = -\sigma_{rr} = 0$$

$$t_\theta = -\sigma_{r\theta} = 0$$

$$\therefore \quad 2Aa - \frac{2B}{a^3} + \frac{D}{a} = 0 \qquad \text{— (1)}$$

at $r=b$, traction free:

$$\therefore \quad 2Ab - \frac{2B}{b^3} + \frac{D}{b} = 0 \qquad \text{— (2)}$$




@ $\theta = 0$,

$t_r = \sigma_{rr}(0) + \sigma_{r\theta}(1) = \sigma_{r\theta}$

$t_\theta = \sigma_{\theta\theta}$

$$F_\theta = \int_a^b \sigma_{\theta\theta}\, dr = \int_a^b \sigma_{\theta\theta}(r,\theta)\, d\theta = T$$

$$3A(b^2-a^2) + \frac{B}{a^2b^2}(b^2-a^2) + D\log\left(\frac{b}{a}\right) = T \quad \text{— (3)}$$

3 Eq$^{n}$s $\longrightarrow$ 3 unknowns

$$A = \frac{T}{2N}, \quad B = -\frac{Ta^2b^2}{2N}, \quad D = -\frac{T(a^2+b^2)}{N}$$

where $N = (b^2-a^2) - (a^2+b^2)\log\frac{b}{a}$

check other boundaries :-

@ $\theta = 0$ :-

$$M = \int_a^b \sigma_{\theta\theta} \cdot r\, dr \Longrightarrow$$

solving :- $M = \int_a^b \sigma_{\theta\theta}\, r\, dr = 0$

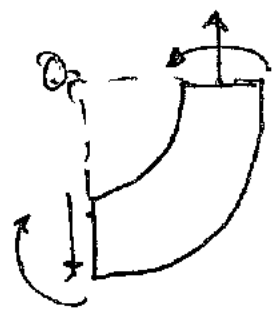
This is incorrect.

We need :- $\int_a^b \sigma_{\theta\theta}\, r\, dr = M + T\frac{(a+b)}{2}$ ✗

$$F_r = \int_a^b \sigma_{r\theta}\, dr = 0 \quad \checkmark$$

@ $\theta = \pi/2$.

$$F_\theta = \int_a^b \sigma_{\theta\theta}\, dr = 0 \quad \checkmark$$



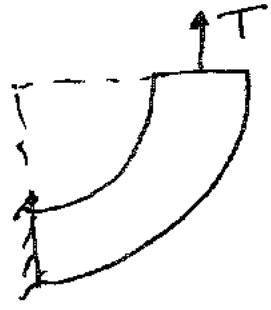

$$M = \int_a^b \sigma_{\theta\theta} \cdot r\, dr = 0 \quad \times$$

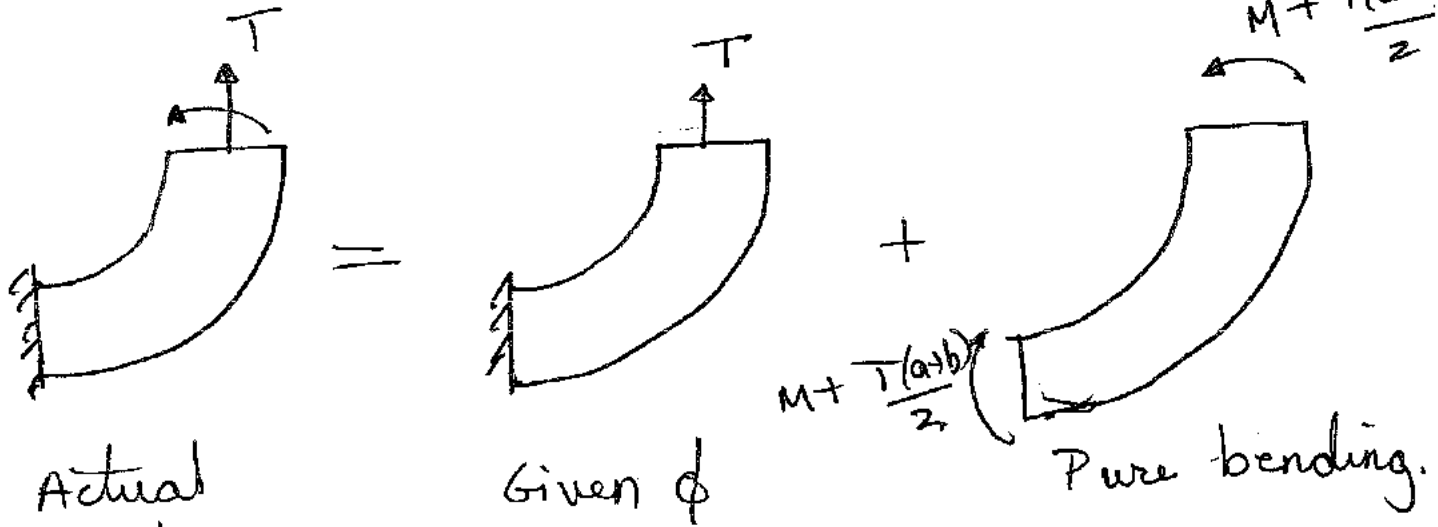
we need :- $\int_a^b \sigma_{\theta\theta}\, r\, dr = M + T\frac{(a+b)}{2}$

$$F_r = \int_a^b \sigma_{r\theta}\, dr = T \quad \checkmark$$

Thus, the moments at $\theta = 0$ & $\theta = \pi/2$ are not satisfied.

The problem solved by given $\phi$ is




Actual Prob

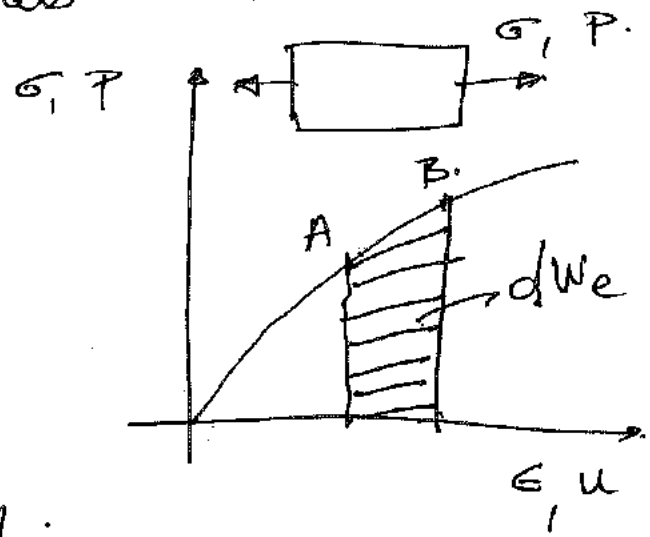
Given $\phi$

Pure bending.

Exam 2: Chapters 4 & 5 (Lect 15 - 27)

Dec 15th: 3:00 – 4:50 PM (WH 210).

## Chapter 4:- Elastic materials




Work done :-

$$dW_e = \int_V \sigma_{ij}\, d\epsilon_{ij}\, dV$$

$\sigma_{ij}$ → Cauchy stress

$\epsilon_{ij}$ → infinitesimal strain.

Work done = strain energy (stored in the body).

$$dW_e = dU = \int_V \sigma_{ij}\, d\epsilon_{ij}\, dV = \int_V d\bar{u}\, dV$$

$$d\bar{u} = \sigma_{ij}\, d\epsilon_{ij} = \text{strain energy density}$$

In undeformed configuration:-

$$dW_e = dU = \int_{V^0} \sigma_{ij}^{K}\, dE_{ij}\, dV^0$$

$\sigma_{ij}/\epsilon_{ij}$ energetically equal to $\sigma_{ij}^{K}/E_{ij}$

- Material objectivity (Frame indifference).

Phenomena remains unchanged even if we change our reference frame.

Tensors that are frame indifferent are called objective tensors.

$\sigma_{ij}^{*}$ & $E_{ij}$ → objective. So we use them to formulate material behavior.

- Elastic material:

$$\sigma_{ij}^{K} = G_{ij}(E_{kl})$$

$$\sigma_{ij} = f_{ij}(\epsilon_{kl})$$

For large deformations:

$$\sigma_{ij}^{K} = \frac{\partial \bar{U}}{\partial \epsilon_{ij}} = 2\frac{\partial \bar{U}}{\partial C_{ij}} \Rightarrow C_{ij} = 2E_{ij} + \delta_{ij}$$

- Small strain elastic materials.

$$\sigma_{ij} = f_{ij}(\epsilon_{kl})$$

$$\sigma_{ij} = f_{ij}\epsilon_{kl}\Big|_{\epsilon_{kl}=0} + \frac{\partial f_{ij}}{\partial \epsilon_{kl}}\Big|_{\epsilon_{kl}=0}\epsilon_{kl} + \frac{\partial^2 f_{ij}}{\partial \epsilon_{kl}\partial \epsilon_{mn}}\Big|_{\substack{\epsilon_{kl}=0 \\ \epsilon_{mn}=0}}\epsilon_{kl}\epsilon_{mn} + \cdots$$

$f_{ij}\epsilon_{kl}\Big|_{\epsilon_{kl}=0}$ → Residual stress

$\frac{\partial^2 f_{ij}}{\partial \epsilon_{kl}\partial \epsilon_{mn}}\Big|_{\epsilon_{kl}=0}$ → Physical non-linearity

Geometric non-linearity ( NLGEOM (Abaqus)

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} - \frac{\partial u_m}{\partial x_i}\frac{\partial u_m}{\partial x_j}\right)$$

↑ Geometric nonlinearity

- Linear Elasticity : $\sigma_{ij} = \frac{\partial f_{ij}(0)}{\partial \epsilon_{kl}} \epsilon_{kl}$

$$\sigma_{ij} = C_{ijkl}\,\epsilon_{kl}$$

↑ Elastic constants.

- Voigt Notation : $\{\sigma\} = [C]\{\epsilon\}$

$$\{\sigma\} = \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} \qquad \{\epsilon\} = \begin{Bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ 2\epsilon_{23} \\ 2\epsilon_{13} \\ 2\epsilon_{12} \end{Bmatrix}$$

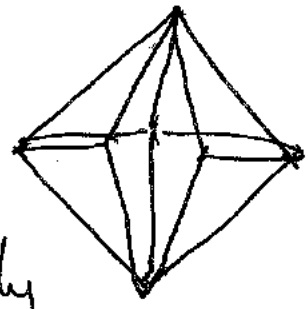
- Material symmetries

(1) Monoclinic system — One symmetry plane.
13 independent elastic constants

② Orthotropic system (Triclinic).

3 mutually orthogonal planes of symmetry

9 independent constants.



③ Hexagonal system (Transversely isotropic).

1 plane of symmetry

+ 1 axis of symmetry $\perp^{or}$ to plane of symmetry.

↳ 5 independent constants.

④ Isotropic systems → Independent of direction.

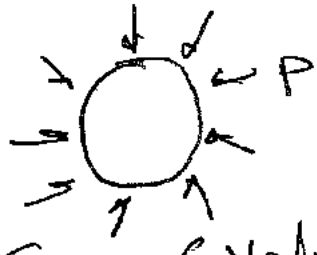
2 independent constants. $(E, G)$ $(\lambda, \mu)$

$$\sigma_{ij} = 2\mu \varepsilon_{ij} + \lambda \varepsilon_{kk} \delta_{ij}$$

$$2\mu \varepsilon_{ij} = \sigma_{ij} - \frac{\lambda}{3\lambda + 2\mu} \sigma_{kk} \delta_{ij}$$

– Loading examples :–

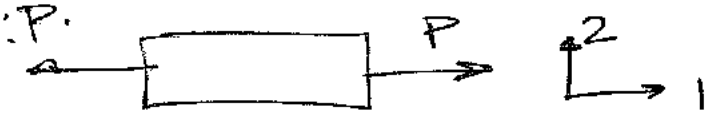
① Hydrostatic pressure :




$$\sigma_{ij} = -p \delta_{ij}$$

Bulk modulus : $K = \frac{1}{3} \frac{\sigma_{kk}}{\varepsilon_{kk}}$ (Volume change)

② Simple Tension : P.




Elastic modulus $E = \frac{\sigma_{11}}{\epsilon_{11}}$

Poisson's ratio : $\vartheta = -\frac{\epsilon_{22}}{\epsilon_{11}}$

③ Simple shear

Shear modulus $G = \frac{\sigma_{12}}{2\epsilon_{12}} = \mu$

— Chapter 5 : Linear Elasticity Problems.

① Linear (Classical) elasticity
- Small deformations
- Linear, isotropic material
- No inertial effects

Then :-

$$\left.\begin{aligned} &\frac{\partial \sigma_{ij}}{\partial x_j} + \rho b_i = 0 \\ &\sigma_{ij} = \sigma_{ji} \\ &\epsilon_{ij} = \frac{1}{2}\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right] \\ &\sigma_{ij} = 2\mu\epsilon_{ij} + \lambda\epsilon_{kk}\delta_{ij} \end{aligned}\right\} \begin{aligned} &15\ \text{Eq} \\ &15\ \text{unknowns} \end{aligned}$$

specify $u_i$ or $t_i = \sigma_{ij}n_j$ on boundaries

- Displacement formulation:

  3 Equations, 3 unknowns.

- Stress formulation:

  6 unknowns.

→ Compatibility :- $2\frac{\partial^2 \epsilon_{xy}}{\partial x \partial y} = \frac{\partial^2 \epsilon_{xx}}{\partial y^2} + \frac{\partial^2 \epsilon_{yy}}{\partial x^2}$

* Solving a problem should result in a stress field that satisfies :-

  (1) Equilibrium
  (2) Compatible
  (3) Boundary conditions.

- Plane problems:

  (1) Plane strain :- $u_3 = 0$

  $\epsilon_{33} = 0$, $\sigma_{13} = 0$, $\epsilon_{23} = 0$

  (2) Plane stress : $\sigma_{13} = \sigma_{23} = \sigma_{33} = 0$

  ↳ approximate

- Airy stress function; (2D).

  A function $\phi$, such that

$$\sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2}, \quad \sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2}, \quad \sigma_{xy} = \frac{-\partial^2 \phi}{\partial x \partial y}$$

  ↳ Equilibrium satisfied.

$\nabla^4 \phi = 0$ (compatibility).

$\phi \rightarrow$ what T.S.C does it satisfy?

If $\phi$ satisfies these conditions, then it solves an elasticity problem.